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Regressive partition relations, n-subtle cardinals, and Borel diagonalization

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Abstract

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We consider natural strengthenings of H. Friedman's Borel diagonalization propositions and characterize their consistency strengths in terms of the n-subtle cardinals. After providing a systematic survey of regressive partition relations and their use in recent independence results, we characterize n-subtlety in terms of such relations requiring only a finite homogeneous set, and then apply this characterization to extend previous arguments to handle the new Borel diagonalization propositions.

In previous papers [6, 7] we showed how regressive partition relations provide a simplifying and unifying scheme for establishing the independence of the Paris–Harrington as well as the Friedman [3] propositions. In these contexts the more informative approach of using regressive partition relations to generate indiscernibles in models can replace the abstract diagonalization technique of Cantor and Gödel for substantiating transcendence. Friedman's proposition correlated with the n-Mahlo cardinals. Here we show how the regressive partition formulation leads directly to an extension that correlates with the n-subtle cardinals, far stronger in consistency strength. In Section 1 we provide a systematic survey of regressive partition relations, their use in independence results, and related open questions. In Section 2 we establish a regressive partition result about n-subtle cardinals, and finally in Section 3 we use it to motivate and characterize the aforementioned extension.

1. Regressive partition relations

Let X be a set of ordinals and $n \in \omega$. If f is a function with domain $[X]^n$, we write $f(\alpha_0, \ldots, \alpha_{n-1})$ for $f(\{\alpha_0, \ldots, \alpha_{n-1}\})$, with the understanding that

$\alpha_0 < \cdots < \alpha_{n-1}$. Such a function is called *regressive* iff $f(\alpha_0, \ldots, \alpha_{n-1}) < \alpha_0$ whenever $\alpha_0 < \cdots < \alpha_{n-1}$ all belong to X and $\alpha_0 > 0$. There is a natural notion of homogeneity for such a function f: $Y \subseteq X$ is *min-homogeneous for* f iff whenever $\alpha_0 < \cdots < \alpha_{n-1}$ and $\beta_0 < \cdots < \beta_{n-1}$ all belong to Y, $\alpha_0 = \beta_0$ implies $f(\alpha_0, \ldots, \alpha_{n-1}) = f(\beta_0, \ldots, \beta_{n-1})$. In other words, f on an n-tuple from Y depends only on the first element. We write $X \to (\gamma)^n_{\text{reg}}$ iff whenever f on $[X]^n$ is regressive, there is an $Y \in [X]^\gamma$ min-homogeneous for f.

If the usual partition relation emanating from Ramsey's Theorem can be viewed as a generalization of the Pigeon-Hole Principle, then the regressive partition relation can be regarded as a generalization of Fodor's Theorem on regressive functions on stationary sets. The relation is actually a special case of the *canonical* partition relation of Erdös–Rado [2]. The following is an immediate consequence of their canonical generalization of Ramsey's Theorem:

For any $n \in \omega$, $\quad \omega \to (\omega)^n_{\text{reg}}$.

In Kanamori–McAloon [7] the direct "minaturization" of this proposition,

(*) For any $n, k \in \omega$ there is an $m \in \omega$ such that $m \to (k)^n_{\text{reg}}$,

is shown to be equivalent to the well-known Paris–Harrington [8] proposition and hence unprovable (in a strong sense) in Peano Arithmetic. In fact, it is shown that (*) for fixed n is equivalent to Paris–Harrington for fixed n and hence unprovable in $\mathrm{I}\Sigma_{n-1}$, induction restricted to Σ_{n-1} formulas. The transparent independence proofs in [7], which quickly provide indiscernibles for models, argue for the efficacy of regressive partitions in this context. We mention here two open questions:

1.1. Question. Is the following proposition independent of Peano Arithmetic?: For any $n \in \omega$ there is an $m \in \omega$ such that $m \to (n+2)^n_{\text{reg}}$.

The $n+2$ here is the minimal value for a non-trivial partition relation and allows very little flexibility; the [7] independence proof ostensibly needs $2n$ (or rn for any real $r > 1$) in place of $n+2$. There is an analogous open question concerning the Paris–Harrington proposition.

In the context of the Friedman–Simpson Reverse Mathematics program, it has been observed that over the base theory RCA_0 (Recursive Comprehension Axiom), the system ACA_0 (Arithmetical Comprehension Axiom) is equivalent to the system axiomatized by $\omega \to (\omega)^3_2$. It is not known whether the superscript 3 can be replaced by 2. This is the so-called "3–2 Problem", another problem about minimal hypotheses for transcendence, and has the following recursion-theoretic formulation:

1.2. Question. Is there a recursive map $f: [\omega]^2 \to 2$ such that for any infinite homogeneous H, $0' \leqslant_{\mathrm{T}} H$?

Clote has observed that over RCA_0, the system axiomatized by $\omega \to (\omega)^2_{\text{reg}}$ is equivalent to ACA_0, so that the exponent *can* be lowered if regressive partitions are used.

Turning to the infinite case, we already mentioned the Erdös–Rado result for ω. To get min-homogeneous sets of size $\kappa > \omega$ with exponent 2, $\lambda \to (\kappa)^2_{\text{reg}}$, simple arguments show that λ can be taken accessible from κ as in the familiar Erdös–Rado Theorem for ordinary partition relations. However, it turns out that for exponents $\geqslant 3$ regressive partition relations provide characterizations of the n-Mahlo cardinals for $n \in \omega$, as was first established by Schmerl. Recall that the n-Mahlo cardinals are the least large cardinals conceptually transcending inaccessibility: κ is 0-Mahlo iff κ is (strongly) inaccessible; and κ is $n+1$-Mahlo iff every closed unbounded subset of κ contains an n-Mahlo cardinal. The following was established by Schmerl in a different notation:

1.3. Theorem (Schmerl [9]). *The following are equivalent for cardinals $\kappa > \omega$ and $n \in \omega$:*

(a) *κ is n-Mahlo.*

(b) *For any $m \in \omega$ and unbounded $X \subseteq \kappa$, $X \to (m)^{n+3}_{\text{reg}}$.*

(c) *For any unbounded $X \subseteq \kappa$, $X \to (n+5)^{n+3}_{\text{reg}}$.*

Rather unexpectedly, a partition relation for $\kappa > \omega$ only requiring a finite homogeneous set characterizes a large cardinal. This idea is pursued in useful form for the n-subtle cardinals in Section 2.

The following theorem completed the characterization of regressive partition relations.

1.4. Theorem (Schmerl [9] for (b), Hajnal–Kanamori–Shelah [5] for (c)). *The following are equivalent for cardinals $\kappa > \omega$ and $0 < n < \omega$:*

(a) *κ is n-Mahlo.*

(b) *For any $\gamma < \kappa$ and unbounded $X \subseteq \kappa$, $X \to (\gamma)^{n+2}_{\text{reg}}$.*

(c) *For any closed unbounded $C \subseteq \kappa$, $C \to (\omega)^{n+2}_{\text{reg}}$.*

Although the partition relation is preserved upon increasing the set on the left, imposing conditions on all unbounded $X \subseteq \kappa$ enables one to have characterizations *at* κ. Keeping this in mind, Theorems 1.3 and 1.4 show how one works one's way up through the regressive partition relation for exponents $n \geqslant 3$: Getting non-trivial min-homogeneous sets of size $m < \omega$ for $n = 3$ requires inaccessibility. Suddenly, getting one of size ω for $n = 3$ requires a 1-Mahlo cardinal κ. Moreover, we can then get min-homogeneous sets of any size $< \kappa$ for $n = 3$, as well as of any size $< \omega$ for $n = 4$. Repeating the pattern, to get a min-homogeneous set of size ω for $n = 4$ requires a 2-Mahlo, and so forth.

We next discuss the interplay between these characterizations of n-Mahlo cardinals and a "Borel diagonalization" proposition of Friedman [3]. He formulated and investigated several rather concrete propositions about Borel measurable functions (and in later work about spaces of groups and the like, and finite propositions—see Stanley [10] and Friedman [4]) which turned out through clever coding to have remarkably strong consistency strengths in terms of large cardinal hypotheses in set theory.

To recapitulate some notation and concepts, let I be the unit interval of reals and $Q = {}^{\omega}I$ (the Hilbert cube) the set of countable sequences drawn from I. If $n \in \omega$ and $y, z \in {}^{n}Q$, say that $y \sim z$ iff there is a permutation ρ of ω, which is the identity except at finitely many arguments, such that $y(i) \circ \rho = z(i)$ for each $i < n$. Let us say that a function F with domain ${}^{n}Q$ is *totally invariant* iff whenever $y, z \in {}^{n}Q$ and $y \sim z$, then $F(y) = F(z)$. A function G with domain $Q \times {}^{n}Q$ is *right-invariant* iff whenever $x \in Q$, $y, z \in {}^{n}Q$, and $y \sim z$, then $G(x, y) = G(x, z)$.

Friedman's proposition P from his [3] is $\forall n \in \omega\, P_n$, where

(P_n) Suppose $F: Q \times {}^{n}Q \to I$ is Borel and right-invariant. Then for any $m \in \omega$ there is a sequence $\langle x_k \mid k < m \rangle$ of distinct elements of Q such that: whenever $s < t_1 < \cdots < t_n < m$, $F(x_s, \langle x_{t_1}, \ldots, x_{t_n} \rangle)$ is the first coordinate of x_{s+1}.

Note the analogy between the conclusion and min-homogeneity. Friedman motivated P as a sequential generalization of a basic Borel diagonalization proposition that he established in ZFC:

> If $F: Q \to I$ is Borel and totally invariant, then there is an $x \in Q$ such that $F(x) \in$ the range of x.

This was in turn motivated by Cantor's original topological proof that I is not countable, which amounted to showing that "totally invariant" cannot be dropped from above. As Friedman emphasized, "Borel" can be replaced by "finitely Borel", i.e. of a finite rank in the Baire hierarchy, without affecting the strength of P and thus bringing it into the fold of "concrete" mathematics. In particular, unlike other propositions like Suslin's Hypothesis, P is absolute with respect to relativization to the constructible universe L.

Friedman established:

1.5. Theorem (Friedman [3]). *The following are equivalent*:

(a) *P(even just for finitely Borel functions).*

(b) *For any $a \subseteq \omega$ and $n \in \omega$, there is an ω-model containing a of* ZFC + $\exists\kappa$ (κ *is n-Mahlo*).

In the forward direction, P_{n+4} is used with an appropriate right-invariant Borel function to generate a finite sequence of reals that corresponds to a set of indiscernible ordinals in a "min" sense in an ω-model of ZFC + $V = L$. The characterization 1.3 is then invoked to show that there is a n-Mahlo cardinal in the model. In the converse direction, given a Borel function F as hypothesized in P_n, one works with a countable ω-model containing an $a \subseteq \omega$ coding F of ZFC + $\exists\kappa$ (κ is n-Mahlo). In the Levy collapse of an n-Mahlo cardinal κ, ordinals $<\kappa$ are associated with members of Q, and Theorem 1.3 is used with a function based on corresponding forcing terms to verify P_n.

Kanamori [6] refined the proof of Theorem 1.5 and developed more technical propositions $\bar{P}_n$ for $n \in \omega$ to provide near equivalences for a level-by-level analysis:

1.6. Theorem (Kanamori [6]). *For any $n \in \omega$:*

(a) *If $\bar{P}_{n+2}$ holds (even just for Borel functions of rank <3), then for any $a \subseteq \omega$ there is an ω-model containing a of* ZFC + $\exists\kappa$ (κ *is n-Mahlo*).

(b) *If for any $a \subseteq \omega$ there is an ω-model containing a of* ZFC + $\exists\kappa\, \exists\delta > \kappa$ (κ *is n-Mahlo and $L_\kappa[a] < L_\delta[a]$), then $\bar{P}_{n+2}$ holds.*

This was motivated by a question of Friedman, reminiscent of Questions 1.1 and 1.2 in minimizing hypotheses, that remains unresolved:

1.7. Question. Is P_3 independent of ZFC?

P_2 may also be independent, with the overall scheme suggesting that it may entail the existence of an ω-model of ZFC + $\exists\kappa$ (κ is inaccessible).

The refinement of Theorem 1.6 over Theorem 1.5 is based to a large extent on a succinct extension of Theorem 1.3:

1.8. Theorem (Kanamori [6]). *Suppose that $n \in \omega$ and X is a set of ordinals such that $X \cap \omega = \emptyset$. Then $X \to (n+5)^{n+3}_{\mathrm{reg}}$ iff $X \cap \kappa$ is unbounded in κ for some n-Mahlo cardinal κ.*

The point is that the regressive partition relation for a *single* set X requiring only a finite min-homogeneous entails the existence of n-Mahlo cardinals. $X \cap \omega = \emptyset$ corresponds to the $\kappa > \omega$ case in Theorem 1.3, avoiding the known cases $\leqslant\omega$. Theorem 1.8 inspired an analogous assertion about n-subtle cardinals, which is the crucial ingredient in the extension of P in Section 3.

2. *n*-subtle cardinals

The n-subtle cardinals were introduced by Baumgartner [1] as generalizations of the subtle cardinals, isolated by Jensen and Kunen in their investigation of combinatorial principles in L. Compatible with $V = L$, the cardinals chart the territory between the weakly compact cardinals and the existence of $0^\#$ in the hierarchy of large cardinal hypotheses in set theory. Through a combinatorial analysis of their incipient definitions, Baumgartner provided regressive partition characterizations to which Theorem 1.3 bears an evident relation.

For X a set of ordinals and $n \in \omega$, we write $X \to (\gamma)^n_{\mathrm{regr}}$ iff whenever f on $[X]^n$ is regressive, there is a $Y \in [X]^\gamma$ homogeneous for f (in the usual sense). Requiring homogeneous rather than just min-homogeneous sets turns out to be a considerable strengthening. For present purposes, we can comprehend the n-subtle cardinals through the following characterization.

2.1. Theorem (Baumgartner [1]). *Suppose that $0 < n < \omega$. Then the following are equivalent for a cardinal κ:*

(a) *κ is n-subtle.*

(b) *For any closed unbounded $C \subseteq \kappa$, $C \to (n+2)^{n+1}_{\text{regr}}$.*

(c) *For any closed unbounded $C \subseteq \kappa$, and f regressive on $[C]^{n+1}$, there is an inaccessible cardinal $\lambda < \kappa$ and an unbounded $Y \subseteq \lambda$ homogeneous for f.*

The following is the needed analogue of Theorem 1.8; although the proof is similar, we include it because of the subtle differences.

2.2. Theorem. *Suppose that $0 < n < \omega$ and X is a set of ordinals such that $X \cap 2 = \emptyset$. If $X \to (n+2)^{n+1}_{\text{regr}}$, then $X \cap \kappa$ is unbounded in κ for some n-subtle cardinal κ.*

Unlike Theorem 1.8 this is not an equivalence, with X the set of successor ordinals below an n-subtle cardinal being a counterexample. The refinement to $X \cap 2 = \emptyset$ is the natural one in the present context, but the first lemma toward the theorem provides a more useful condition.

2.3. Lemma. *Under the assumptions of Theorem 2.2, $X \to (\gamma)^n_{\text{regr}}$ iff $X \sim \omega \to (\gamma)^n_{\text{regr}}$.*

Proof. In the non-trivial direction, suppose that f is regressive on $[X \sim \omega]^n$; we must find an $Y \in [X \sim \omega]^\gamma$ homogeneous for f. Define g on $[X]^n$ as follows:

$$g(s) = \begin{cases} f(s) & \text{if } s \cap \omega = \emptyset, \\ \min(s) - 1 & \text{if } s \subseteq \omega, \text{ else} \\ 0 & \text{if } |s \cap \omega| \text{ is even}, \\ 1 & \text{if } |s \cap \omega| \text{ is odd}. \end{cases}$$

g is regressive since $X \cap 2 = \emptyset$, so by hypothesis there is an $Y \in [X]^\gamma$ homogeneous for g. We can, of course, assume that $\gamma > n$. If $Y \subseteq \omega$, we can easily derive a contradiction using the second clause of g. If $Y \cap \omega \neq \emptyset$ and $Y \sim \omega \neq \emptyset$, then we can easily derive a contradiction using the third and fourth clauses of g. Hence, $Y \cap \omega = \emptyset$, and we are done. $\square$

The following lemma contains the crux of Theorem 2.2.

2.4. Lemma. *Suppose that $2 \leq n < \omega$ and for some limit ordinal η, and C and X are subsets of $\eta \sim \omega$ with C closed unbounded. If $C \not\to (\gamma)^n_{\text{regr}}$ and $X \cap \xi \not\to (\gamma)^n_{\text{regr}}$ for every $\xi < \eta$, then $X \not\to (\gamma)^n_{\text{regr}}$.*

Proof. We first handle the cases $n \geq 3$. Set $\bar{C} = C \cup \{\omega\}$. For each $\alpha \in X$ set $\psi(\alpha) = \sup(\bar{C} \cap (\alpha + 1))$, an element of $\bar{C}$ since C is closed unbounded and

$\min(X) \geq \omega$. We first define the *type* of a member of $[X]^n$ according to $\bar{C}$ as follows: If $\alpha_0 < \cdots < \alpha_{n-1}$ all belong to X, let $\langle \xi_0, \ldots, \xi_k \rangle$ enumerate the set $\{\psi(\alpha_i) \mid i < n\}$ in increasing order, and set $r_j = |\{i \mid \psi(\alpha_i) = \xi_j\}|$ for $j \leq k$. Then the *type* of $\{\alpha_0, \ldots, \alpha_{n-1}\}$ is $\langle r_0, \ldots, r_k \rangle$, which we can assume through sequence coding is a natural number $\neq 1$.

Next, let g attest to $C \not\rightarrow (\gamma)^n_{\text{regr}}$ and g_ξ attest to $X \cap \xi \not\rightarrow (\gamma)^n_{\text{regr}}$ for $\xi < \eta$. Since $C, X \subseteq \eta \sim \omega$, we can assume through renumbering that the ranges of g and of the g_ξ's do not contain 1 or any number coding a type. Now define G on $[X]^n$ as follows:

$$G(\alpha_0, \ldots, \alpha_{n-1}) = \begin{cases} 1 & \text{if } \omega = \psi(\alpha_0) < \cdots < \psi(\alpha_{n-1}), \\ g(\psi(\alpha_0), \ldots, \psi(\alpha_{n-1})) & \text{if } \omega < \psi(\alpha_0) < \cdots < \psi(\alpha_{n-1}), \\ g_\xi(\alpha_0, \ldots, \alpha_{n-1}) & \text{if } \psi(\alpha_0) = \cdots = \psi(\alpha_{n-1}), \\ & \quad \text{where } \xi \text{ is the next element of} \\ & \quad C \text{ after } \psi(\alpha_1), \\ \text{type of } \{\alpha_0, \ldots, \alpha_{n-1}\} & \text{otherwise.} \end{cases}$$

G is regressive, so suppose that $Y \subseteq X$ is homogeneous for G. We can assume that Y has at least $n+1$ elements.

Using $n \geq 3$ and the last clause of G, it is simple to see that ψ must be either constant or injective on Y. If ψ is constant on Y, then Y cannot have ordertype γ by the third clause of G. If ψ is injective on Y, then $\psi(\min(Y)) \neq \omega$ by the first clause of G. In this case, Y cannot have ordertype γ by the second clause of G. This completes the proof for $n \geq 3$.

In the special case $n = 2$, for every infinite ordinal ρ let f^{e}_ρ be a bijection between ρ and the *even* ordinals less than ρ, and f^{o}_ρ a bijection between ρ and the *odd* ordinals $\neq 1$ less than ρ. Now define G on $[X]^2$ as before, but modulated by these functions:

$$G(\alpha_0, \alpha_1) = \begin{cases} 1 & \text{if } \omega = \psi(\alpha_0) < \psi(\alpha_1), \\ f^{\text{e}}_{\psi(\alpha_0)}(g(\psi(\alpha_0), \psi(\alpha_1))) & \text{if } \omega < \psi(\alpha_0) < \psi(\alpha_1), \\ f^{\text{o}}_{\alpha_0}(g_\xi(\alpha_0, \alpha_1)) & \text{if } \psi(\alpha_0) = \psi(\alpha_1). \end{cases}$$

We can now argue as before, discerning cases by whether the constant value on the homogeneous set is even or odd. The proof of Lemma 2.4 is complete. □

Proof of Theorem 2.2. With the given hypotheses, let η be the least ordinal such that $X \cap \eta \rightarrow (n+2)^{n+1}_{\text{regr}}$. Then $\eta > \omega$ by a simple argument. By Lemma 2.3 we can assume that $X \cap \omega = \emptyset$. But then Lemma 2.4 implies that for any closed unbounded $C \subseteq \eta$, $C \rightarrow (n+2)^{n+1}_{\text{regr}}$, i.e. η is n-subtle. □

3. The proposition H

The clear analogy between Theorems 1.8 and 2.2 and the formulation and analysis in [6] of $\bar{P}_n$ (which we did not bother to state here) leads to the following extension of P:

$$H \quad \text{is} \quad \forall n\, H_n,$$

where

(H_n) Suppose that $F_1 : Q \times {}^nQ \to I$ and $F_2 : Q \times Q \to I$ are Borel and right-invariant such that $F_1(x, y) \in$ the range of x for $x \in Q$ and $y \in {}^nQ$. Then for any $m \in \omega$ there is a sequence $\langle x_i \mid i < m \rangle$ of distinct elements of Q such that:
(a) if $s < t < m$, then $F_2(x_s, x_t)$ is the first coordinate of x_{s+1},
and
(b) $\langle x_i \mid i < m \rangle$ is homogeneous for F_1.

Here, of course, "homogeneous" means that $F(x_s, \langle x_{t_1}, \ldots, x_{t_n} \rangle)$ is independent of the choice of $s < t_1 < \cdots < t_n < m$. The lack of dependence even on s corresponds to the move to homogeneity from min-homogeneity, and the "choice function" condition $F_1(x, y) \in$ range of x corresponds to regressiveness. The diagonalization analogy between H_n and P_n is maintained in the use of F_2 and (a) which figure in the proof below; from an esthetic point of view, eliminating them is desirable, and may be possible with a more subtle analysis.

The following is the main theorem of the paper. Because of the analogy established between the n-Mahlo and n-subtle cardinals, the proof amounts to a modification of the proof of Theorem 1.5. Consequently, we only provide details on the amendments, based in the forward direction on the approach of [6].

3.1. Theorem. *The following are equivalent*:
(a) H (*even just for Borel functions of rank* <3).
(b) *For any* $a \subseteq \omega$ *and* $0 < n < \omega$, *there is an* ω-*model containing* a *of* ZFC $+ \exists\kappa$ (κ *is* n-*subtle*).

Remark. A level-by-level analysis along the lines of 1.6 is presumably possible at the cost of developing more technical propositions akin to $\bar{P}_n$; a finer proof than the one below would then have to be developed.

Proof. (a)$\to$(b) As in [3], let $\mathscr{L}$ be the language of second-order arithmetic augmented by class variables for *subsets* of $\mathscr{P}(\omega)$ (but no quantifiers for these variables). Any $A \subseteq \mathscr{P}(\omega)$ is regarded as a structure for $\mathscr{L}$ in the natural way, with first-order variables ranging over members of A that happen to be integers, second-order variables ranging over members of A, and class variables ranging over arbitrary subsets of A. A formula ψ of $\mathscr{L}$ is Σ^1_k if it has $k-1$ alternations of second-order quantifiers beginning with an existential quantifier, followed by only bounded first-order quantifiers. For $x \subseteq \omega$ let $|x| = \{\{m \mid 2^n 3^m \in x\} \mid n \in w\} \subseteq \mathscr{P}(\omega)$. Modifying Friedman's notion of (n, k)-critical sequence, say that $\langle x_i \mid i < d\rangle$ for $d \in \omega$ is an *n-subline* sequence iff each $x_i \subseteq \omega$ and:

(i) for all $s < t < d$ and all Σ^1_2 formulas ψ we have $x_s \in |x_t|$ and $\{j \in \omega \mid |x_t| \models \psi(j, x_s)\} \in |x_{s+1}|$, and

(ii) for all $t_1 < \cdots < t_n < d$, $u_1 < \cdots < u_n < d$, $a \in |x_{\min\{t_1, u_1\}}|$ and Σ^1_2 formulas ψ in a finite collection Ψ (described below), we have

$$|x_{t_{n+1}}| \models \psi(a, |x_{t_1}|, \ldots, |x_{t_n}|) \quad \text{iff} \quad |x_{u_{n+1}}| \models \psi(a, |x_{u_1}|, \ldots, |x_{u_n}|).$$

Here, Ψ is a finite collection of Σ^1_2 formulas which can be determined *a priori*, so that the above indiscernibility property for these formulas suffices to push through the main argument below for (a)$\to$(b) of the theorem. The use of Σ^1_2 formulas follows [6], and is the reason why we can restrict H to Borel functions of rank <3.

3.2. Lemma. *If $n > 0$ and H_{n+1} holds, then for any $d \in \omega$ there is an n-sublime sequence of length d.*

Proof. To apply H_{n+1}, we make the natural switch from I to $\mathscr{P}(\omega)$. For $x \in {}^\omega\mathscr{P}(\omega)$ let $\bar{x} = \{2^n 3^m \mid m \in x(n)\}$ and $\mathrm{Rng}(x)$ be the range x. For any formula ψ of $\mathscr{L}$ let $\#\psi$ denote its Gödel number in some fixed arithmetization.

Define $F_1 : {}^\omega\mathscr{P}(\omega) \times {}^{n+1}({}^\omega\mathscr{P}(\omega)) \to \mathscr{P}(\omega)$ and $g : {}^{n+2}({}^\omega\mathscr{P}(\omega)) \to \omega \times 3$ as follows: Suppose that $x, x_1, \ldots, x_{n+1} \in {}^\omega\mathscr{P}(\omega)$.

Case I. There is a Σ^1_2 formula $\psi \in \Psi$ as in (ii) above and $a \in \mathrm{Rng}(x)$ such that either

(Ia) $\quad \mathrm{Rng}(x_n) \models \psi(a, \mathrm{Rng}(x), \mathrm{Rng}(x_1), \ldots, \mathrm{Rng}(x_{n-1}))$ and
$\quad \mathrm{Rng}(x_{n+1}) \models \neg\psi(a, \mathrm{Rng}(x_1), \ldots, \mathrm{Rng}(x_n))$,

or

(Ib) $\quad \mathrm{Rng}(x_n) \models \neg\psi(a, \mathrm{Rng}(x), \mathrm{Rng}(x_1), \ldots, \mathrm{Rng}(x_{n-1}))$ and
$\quad \mathrm{Rng}(x_{n+1}) \models \psi(a, \mathrm{Rng}(x_1), \ldots, \mathrm{Rng}(x_n))$.

Then let a_0 be such an a so that $a = x(n)$ with n minimal, and for this a let ψ_0 be a such a ψ with $\#\psi$ minimal. Set

$$F_1(x, \langle x_1, \ldots, x_{n+1}\rangle) = a_0, \quad \text{and}$$

$$g(x, x_1, \ldots, x_{n+1}) = \begin{cases} \langle \#\psi_0, \mathbf{1}\rangle & \text{if (Ia) holds,} \\ \langle \#\psi_0, \mathbf{2}\rangle & \text{if (Ib) holds.} \end{cases}$$

Case II. There is no such ψ. Then set

$$F_1(x, \langle x_1, \ldots, x_{n+1}\rangle) = x(0), \quad \text{and} \quad g(x, x_1, \ldots, x_{n+1}) = \langle 0, 0\rangle.$$

Next, define $F_2: {}^{\omega}\mathcal{P}(\omega) \times {}^{\omega}\mathcal{P}(\omega) \to \mathcal{P}(\omega)$, just as in 3.1 of [6], as follows: Suppose that $x, y \in {}^{\omega}\mathcal{P}(\omega)$.

Case 1. There is a Σ^1_2 formula ϕ such that $\{j \in \omega \mid \mathrm{Rng}(y) \models \phi(j, \bar{x})\} \notin \mathrm{Rng}(y)$. Then let ϕ_0 be such a formula with $\#\phi$ minimal, and set $F_2(x, y) = \{j \in \omega \mid \mathrm{Rng}(y) \models \phi_0(j, \bar{x})\}$.

Case 2. There is no such ϕ. Then set $F_2(x, y) = \{\#\phi \mid \phi$ is Σ^1_2 and $\mathrm{Rng}(y) \models \phi(\bar{x})\}$.

Suppose now that $d \in \omega$ is given, assuming $d \geq n + 4$ for non-triviality. F_1 and F_2 satisfy the hypotheses of H_{n+1} (after the switch from I to $\mathcal{P}(\omega)$), and since Ψ in (ii) of n-sublime is finite, the range of g is finite. Hence, we can first apply H_{n+1} to get a sequence sufficiently long so that, by an application of the Finite Ramsey Theorem we can extract a subsequence $\langle x_i \mid i < d \rangle$ satisfying the conclusions of H_{n+1} *and* so that g is constant on ascendingly indexed $n+2$-tuples drawn from the subsequence.

We can now show that $\langle \bar{x}_i \mid i < d \rangle$ is n-sublime. By the argument for 3.1 of [6] using F_2, clause (i) must be satisfied. It is easy to check that the constant value of g must be $\langle 0, 0 \rangle$, and hence by a straightforward indiscernibility argument clause (ii) must also be satisfied. This completes the proof of Lemma 3.2. □

The rest of the proof of (a)→(b) is just as in [3] and [6]. Starting with a sufficiently long $n+1$-sublime sequence, we can build an ω-model of ZFC with an initial segment of the x_i's in the sequence corresponding to "ordinals" in the model. By Theorem 2.2 and the indiscernibility property (ii) of n-sublime, we can then show that in the model there must be an n-subtle cardinal. It can be checked in the argument of [6] that, indeed, only finitely many Σ^1_2 formulas, which we had anticipated with the collection Ψ, need be involved in (ii) of sublime. Finally, for the precise statement of (b), given any $a \subseteq \omega$ it can be used as a parameter in the Σ^1_2 formulas in the definition of n-sublime so that it will be a member of $|x_1|$.

(b)→(a) In this direction we try to exhibit the main ideas by following [3] as closely as we can, foregoing the refinements of [6], for the benefit of the reader. In particular, we outline the argument with $\sim$ in the definition of "right-invariant" replaced by $\approx$, where $x_0 \approx x_1$ means that x_0 and x_1 have the same range. The distracting modifications for getting the result with $\sim$ are just as in [3].

Toward the verification of H_n $(n > 0)$ and maintaining the switch from I to $\mathcal{P}(\omega)$, suppose that $F_1: {}^{\omega}\mathcal{P}(\omega) \times {}^{n}({}^{\omega}\mathcal{P}(\omega)) \to \mathcal{P}(\omega)$ such that $F_1(x, y) \in$ the range of x and $F_2: {}^{\omega}\mathcal{P}(\omega) \times {}^{\omega}\mathcal{P}(\omega) \to \mathcal{P}(\omega)$ are both Borel and right-invariant. Let $a \subseteq \omega$ code Borel codes for F_1 and F_2, and let M be a countable ω-model containing a of ZFC + "κ is $n+1$-subtle". Let C by the "Levy collapse" forcing notion in M, consisting of finite partial functions $f: \kappa \times \omega \to V^M_\kappa$ such that $f(\alpha, i) \in V^M_\alpha$.

Suppose now that $G \subseteq C$ is generic over M. Define $\bar{G}: \kappa \times \omega \to V^M_\kappa$ by $\bar{G}(\alpha, i) = x$ iff $\exists f \in G\, (f(\alpha, i) = x)$. For $x \in M$, set $E(G, x) = \{k \in \omega \mid \exists f \in G$

$(\langle k, f\rangle \in x)\}$. Finally, for limit ordinals $\delta < \kappa$ define $T(G, \delta) \in {}^{\omega}\mathcal{P}(\omega)$ by $T(G, \delta)(m) = E(G, \bar{G}(\delta, m))$.

The following is Lemma 5.12 of [3] and is established using right-invariance:

3.3. Lemma. *Suppose that $\delta < \delta_1 < \kappa$ are limit ordinals and $f \in C$. Then for any $k \in \omega$,*

$$f \vDash k \in F_2(T(\dot{G}, \delta), T(\dot{G}, \delta_1)) \quad \text{iff} \quad f|((\delta+1) \times \omega) \vDash k \in F_2(T(\dot{G}, \delta), T(\dot{G}, \delta_1)).$$

The analogous assertion holds for F_1.

With the choice function condition $F_1(x, y) \in$ range of x, the values of F_1 are determined by even less of the given condition, and this opens the door to the application of n-subtlety. In what follows, we write $p \parallel \phi$ for p decides ϕ, i.e. $p \Vdash \phi$ or $p \Vdash \neg\phi$.

3.4. Lemma. *Suppose that $\delta < \delta_1 < \cdots < \delta_n < \kappa$ are limit ordinals, $f \in C$, and $k \in \omega$. If*

$$f \parallel k \in F_1(T(\dot{G}, \delta), T(\dot{G}, \delta_1), \ldots, T(\dot{G}, \delta_n)),$$

then there is a $g \leqslant f$ and a $\gamma < \delta$ such that

$$g|(\gamma \times \omega) \parallel k \in F_1(T(\dot{G}, \delta), T(\dot{G}, \delta_1), \ldots, T(\dot{G}, \delta_n)).$$

Proof. Since $F_1(x, y) \in$ range of x, let $g \leqslant f$ be such that

$$g \Vdash z = T(\dot{G}, \delta)(m) = F_1(T(\dot{G}, \delta), T(\dot{G}, \delta_1), \ldots, T(\dot{G}, \delta_n))$$

for some term z and $m \in \omega$. By definition of T and E, we can consider z to be definable from $G \mid (\gamma \times \omega)$ for some $\gamma < \delta$. We now show that $g \mid (\gamma \times \omega) \parallel k \in F_1((T(\dot{G}, \delta), T(\dot{G}, \delta_1), \ldots, T(\dot{G}, \delta_n)))$.

Let $h \leqslant g \mid (\gamma \times \omega)$ be arbitrary, and set $j = \max\{i + 1 \mid \exists\alpha\, (\langle \alpha, i\rangle \in \text{domain of } h)\}$.

Define $\hat{g} \in C$ by

$$\hat{g}(\alpha, i) = \begin{cases} g(\alpha, i) & \text{if } \alpha < \gamma \text{ and } g(\alpha, i) \text{ is defined,} \\ g(\alpha, i+j) & \text{if } \alpha \geqslant \gamma \text{ and } g(\alpha, i) \text{ is defined,} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

By an automorphism argument, it follows that

$$\hat{g} \Vdash z = T(\dot{G}, \delta)(m + j) = F_1(T(\dot{G}, \delta), T(\dot{G}, \delta_1), \ldots, T(\dot{G}, \delta_n)).$$

But clearly $\hat{g}$ and h are compatible, so we are done. □

Continuing now with the main argument, we work in M. Let $C = \{\alpha < \kappa \mid \alpha$ is a strong limit cardinal$\}$. Since n + 1-subtle cardinals are inaccessible, C is a closed unbounded subset of κ. Define a function H on $[C]^{n+2}$ as follows: Suppose that $\delta_0 < \delta_1 < \cdots < \delta_{n+1}$ all belong to C.

Case I. There is an $f \in C$ and a $k \in \omega$ such that either

$$\text{(Ia)} \qquad f \Vdash k \in F_1(T(\dot{G}, \delta_0), T(\dot{G}, \delta_1), \ldots, T(\dot{G}, \delta_n)) \\ \sim F_1(T(\dot{G}, \delta_1), T(\dot{G}, \delta_2), \ldots, T(\dot{G}, \delta_{n+1}))$$

or

$$\text{(Ib)} \qquad f \Vdash k \in F_1(T(\dot{G}, \delta_1), T(\dot{G}, \delta_2), \ldots, T(\dot{G}, \delta_{n+1}) \\ \sim F_1(T(\dot{G}, \delta_0), T(\dot{G}, \delta_1), \ldots, T(\dot{G}, \delta_n)).$$

By Lemma 3.4 we can assume that for some $\gamma < \delta_0$,

$$f \mid (\gamma \times \omega) \parallel k \in F_1(T(\dot{G}, \delta_0), T(\dot{G}, \delta_1), \cdots, T(\dot{G}, \delta_n)).$$

Taking f least possible in some fixed well-ordering and γ least possible for this f, set

$$H(\delta_0, \delta_1, \ldots, \delta_{n+1}) = \begin{cases} \langle f \mid (\gamma \times \omega), 1 \rangle & \text{if (Ia) holds,} \\ \langle f \mid (\gamma \times \omega), 2 \rangle & \text{if (Ib) holds.} \end{cases}$$

Case II. There is no such $f \in C$ and $k \in \omega$. Then set $H(\delta_0, \delta_1, \ldots, \delta_{n+1}) = \langle 0, 0 \rangle$.

We can regard H as a regressive function on $[C]^{n+2}$ through coding, since the number of possibilities for f in Case I is small. Suppose that $Y \subseteq C$ is homogeneous for H. Then assuming that $|Y| \geq n+3$, Case I cannot occur on ascending $n+2$-tuples drawn from Y: If for some $\delta_0 < \cdots < \delta_{n+2}$ all belonging to Y and $0 < i < 2$,

$$H(\delta_0, \ldots, \delta_{n+1}) = \langle g, i \rangle = H(\delta, \ldots, \delta_{n+2}),$$

then g decides $k \in F_1(T(\dot{G}, \delta_1), \ldots, T(\dot{G}, \delta_{n+1}))$ in one way by the second equality, but by the first equality and definition, g is extendible to a condition that decides it in the *other* way. Hence, Case II occurs, and the argument of [3] can now be used to get a homogeneous sequence for F_1.

To further handle F_2, note that by Theorem 2.1(c) we can assume that Y is unbounded in some inaccessible cardinal λ. Hence, Theorem 1.3(b) is more than enough so that with the original [3] argument based on Lemma 3.3, we can extract arbitrarily long finite subsequences of Y min-homogeneous for a regressive function corresponding to F_2 and show that the full conclusion of H_{n+1} can be satisfied. This completes our (indication of) the proof of the main Theorem 3.1. □

We point out that Lemma 3.4 was needed to insure that H can be regarded as regressive on a *closed* unbounded set $\subseteq \kappa$ so that we can extract a homogeneous set by subtlety. The function corresponding to F_2 based on Lemma 3.3 need only be regressive on an unbounded set $\subseteq \lambda$ to extract a min-homogeneous set, so a simpler strategem is available—see the H_1^+ idea at the end of the proof of Theorem A in [6].

As in [6] it is possible to develop an infinitary version of H_n with an $F_1: Q \times {}^{<\omega}Q \rightarrow I$ to get a principle with consistency strength at least that of ZFC + $\exists\kappa\,\forall n$ (κ is n-subtle). However, getting an equiconsistency result seems difficult here.

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