PREFACE

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Most of the chapters of this volume present the historical development of aspects of modern, ZFC set theory as a field of mathematics, and a couple of chapters at the end, categorical logic as a foundation of mathematics. Be that as it may, as part of the *Handbook of the History of Logic* the volume can be construed as the one focusing on *extensions vis-à-vis* the intension vs. extension distinction. That distinction, traditionally attributed to Antoine Arnauld in the 1662 *Logique de Port Royal*, is exemplified by "featherless biped", a conceptualization, and the corresponding extension, the collection of all *homo sapiens sapiens*. However this distinction was historically developed and worked in logic, it was in the mathematical development of set theory that extensions became explicit through the mathematization of infinitary concepts and the further development of mathematics, itself increasingly based on informal concepts of set.

How many points are there on the line? This would seem to be a fundamental, even primordial, question. However, to cast it as a *mathematical* question, underlying concepts would have to be invested with mathematical sense and a way of mathematical thinking provided that makes an answer possible, if not informative. First, the real numbers as representing points on the linear continuum would have to be described precisely and extensionally. A coherent concept of cardinality and cardinal number would have to be developed for infinite mathematical collections. And, the real numbers would have to be enumerated in such a way so as to accommodate this concept of cardinality. Georg Cantor made all of these moves as part of the seminal advances that have led to modern set theory. His Continuum Hypothesis would propose a specific, structured resolution about the size of the continuum in terms of his transfinite numbers, and the continuum problem, whether the Continuum Hypothesis holds or not, would henceforth stimulate the development of set theory as its major outstanding problem.

From Cantor's work would come forth new developments drawing out the intension vs. extension distinction, for analyzing his famous diagonal argument Bertrand Russell came in 1901 to his well-known paradox. Thus full "comprehension" was ruled out, as self-applicability precluded that every intension corresponds to a viable extension. Exercised by the implications for any theory of extensions, Russell the artful dodger eventually developed his theory of types, a complicated formal system of logic for mathematics featuring type distinctions for functional application as well as a further ramification into orders. He thus enshrined non-self-applicability, but in his vying for universality he had to build in encumbrances like the Axiom of Reducibility. Russell's theory stood as a remarkable achievement of mathematical logic into the 1920s, when David Hilbert and Paul Bernays focused attention on the restricted, first-order logic as already the provenance of significant issues and Frank Ramsey directed criticism at the intension aspects of the theory as alien to mathematics.

During this period, set theory was crucially advanced in coordination with emerging mathematical practice. Ernst Zermelo effected a first transmutation of the concept of set with his first, 1908 axiomatization, which made explicit principles like the existence of an infinite set, the power set of any set, and the Axiom of Separation, a natural restricted form of comprehension. That axiom together with the Russell paradox argument established Zermelo's first theory, that the universe is not a set, so that "is a set" has no extension. In the 1920s John von Neumann incorporated the transfinite numbers as sets, his ordinals, drawing in the Axiom of Replacement, and promoted the use of proper classes, like the universe, as intrinsic to a coherent axiomatic framework. In Zermelo's final, 1930 axiomatization, the now-standard set theory ZFC is recognizable.

It was Kurt Gödel's 1930s work on the constructible universe L that launched set theory on an independent course as a distinctive field of mathematics. With L he established the relative consistency of the Continuum Hypothesis, synthesizing what had come before in axiomatic set theory with first-order logic through the infusion of metamathematical methods. Through definability, intensionality became woven into a fully extensionalized context. Gödel himself regarded his work on L as a transfinite extension of Russell's types and rectification of the ramified theory, but in any case set theory would henceforth proceed with a further transmutation of the concept of set as stratified into a cumulative hierarchy.

Having become a theory of extensions *par excellence* in coordination with the mathematization of logic, set theory began its transformation into a sophisticated, autonomous filed of mathematics with Paul Cohen's 1963 creation of forcing, with which he established the independence of the Continuum Hypothesis. Forcing quickly became a general method; a myriad of propositions and models were investigated; and lasting structural incentives were put into place.

The first chapter of this volume describes the historical development of set theory up to this point. Its length and detail, as well as those of the other set-theoretic chapters, speaks to the extent and richness of modern set theory. The chapter on the continuum reaches back to Cantor's original incursions and engagingly proceeds forward to how the arithmetical view of the continuum as a collection of points has led to the investigation of combinatorial structures and cardinalities. The chapter on infinite combinatorics describes, from its early beginnings, the direct investigation of the Cantorian transfinite as extension of number, mainly combinatorial partition properties.

The next several chapters deal with those aspects of modern set theory that came into prominence largely after the great expansion of the subject since 1963. The chapter on large cardinals and forcing brings to the fore the mainstream investigation of "strong axioms of infinity" and their intimate relationship with strong propositions through relative consistency, developed through the complementary methods of forcing and inner models. The articulation and analysis of inner models of set theory has itself become an abiding, expansive subject through the infusion of the "fine structure" definability method and pivotal "covering theorems". The next chapter, written by one of the principals involved, proves an informative account of the development of the subject through its self-fueling initiatives.

The investigation of the "determinacy" of infinite games is the most distinctive and intriguing development of modern set theory, and the deep correlations achieved with large cardinals and inner models the most remarkable and synthetic. The chapter on determinacy provides a detailed history and compendium of results that chronicles the growing interest and then the centrality of the subject.

The chapter on singular cardinals describes the most recent major initiative in modern set theory, singular cardinal combinatorics. A critical point was reached in set theory when singular cardinals, owing to their fast approachability from below, provide the setting and motivation for fresh, combinatorial arguments and sophisticated relative consistency results that have articulated the transfinite landscape in new and distinctive ways.

Are there alternatives to the new-standard ZFC set theory? None has led to a comparable, sustained development, but many have arisen owing to a variety of motivations, and their comparative study have generated interesting mathematical analysis and problems, as described in the chapter on alternate set theories.

The alternate to set theory as such for developing extensions had been, from the time of Russell, to have explicit types. But Russell's original theory was complicated and artificial as a matter of logic, and set-theoretic, extensional thinking become inherent in mathematics. Types would come to the fore again however, as new ingredients within mathematics and as part of system building, with the development of category theory and the emergence of computer science. The objects of a category can be taken to be types and the arrows, mappings between the corresponding types. The chapter on sets, types, and categories provides a sweeping, integrative view through to the working out of the "propositions-as-types" doctrine in Martin-Löf's constructive dependent type theory. The chapter on categorical logic provides a detailed, tightly-knit account of the work of the "Montreal school" of F. William Lawvere, André Joyal, and others. These categorical developments have secured theories of extensions separate from set theory and more directly embeddable into the motivational frameworks of mathematical logic.