MAXIMAL SUBBUNDLES OF RANK 2 VECTOR BUNDLES FOR CODING THEORY SEMINAR

ANGELA VIERLING

1. Introduction to Rank Two Vector Bundles

[All of the following is only the briefest sketch to give the ideas. Important details are glossed over or omitted. For the true story, see references such as [5] or [2].]

We are concerned here with rank two vector bundles over projective curves. Briefly, a vector bundle over a variety C is a variety X with a surjective morphism $\pi : X \to C$ such that for every point $P \in C$, there is an open neighborhood $U \ni P$ with $\pi^{-1}(U) \cong U \times \mathbb{A}^r_k$, where k is the base field. So, in particular, $C \times \mathbb{A}^r$ is a vector bundle, called the trivial bundle of rank r. Vector bundles are, by definition, "locally trivial" – we can always find a cover so that on any open set we just have a trivial bundle. An important thing to know about vector bundles is that if we fix a particular trivializing open cover $\{U_\alpha\}$ we can use the local isomorphisms $\phi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times \mathbb{A}^r_k$ to make linear isomorphisms on intersections

$$\phi_\beta \circ \phi_\alpha^{-1} : \mathbb{A}^r_k \xrightarrow{\sim} \mathbb{A}^r_k.$$

Since we have linear maps, we really have matrices and a vector bundle is determined by "transition matrices."

It turns out that vector bundles of rank 2 over a curve C are are the same as ruled surfaces over curves. What are ruled surfaces? For visualization purposes it perhaps suffices to say that ruled surfaces are just like line bundles, except the fiber over every point of a curve is a $\mathbb{P}^1$ rather than an $\mathbb{A}^1$. So, for example, the hyperboloid of one-sheet that students see in vector calculus is (once we take the projective closure) a ruled surface with two rulings. Why does it make sense that rank two vector bundles are the same as ruled surfaces? Instead of looking at the fiber as a two dimensional vector space, we projectivize that two dimensional vector space, getting a $\mathbb{P}^1$ over the vector space.

Another correspondence that we will want to make use of is the correspondence between rank k vector bundles and locally free sheaves of rank k. Here is the "quick-and-dirty" version of sheaves and of the correspondence. A sheaf is a collection of functions that are defined on open subsets of a scheme. There are technical criteria about how these functions must behave and relate to each other. In the "Child's Garden of Sheaves over Curves," the first sheaf discussed would be the sheaf of regular functions on a curve C, denoted $\mathcal{O}_C$. When we are looking at a particular open $U \subset C$ and considering the regular functions on this U, we denote it by $\mathcal{O}_C(U)$. So $\mathcal{O}_C$ contains not just the functions regular on all of C (that would be $\mathcal{O}_C(C)$, also denoted by $\Gamma(\mathcal{O}_C)$ or by $H^0(C, \mathcal{O}_C)$), but all of the functions defined and regular on any open set.

Date: February 24, 2003.

Now, further on in the "Child's Garden of Sheaves over Curves" we find the sheaves of $\mathcal{O}_C$-modules. For such a sheaf $\mathcal{F}$, on each open U, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module. We could also require that for a particular open cover $\{U_i\}$ of C, we always have that $\mathcal{F}(U_i)$ is a direct sum of k copies of $\mathcal{F}(U_i)$, i.e. there exists an isomorphism ϕ_i as below

$$\phi_i : \mathcal{F}\big|_{U_i} \to \mathcal{O}\big|_{U_i}^k.$$

This is what it means to be a locally free sheaf of rank k. It turns out that given a locally free sheaf $\mathcal{F}$ of rank k we can compose the maps ϕ_i and ϕ_j on $U_i \cap U_j$ and get an endomorphism of free sheaves

$$\phi_j \circ \phi_i^{-1} : \mathcal{O}\big|_{U_{ij}}^k \to \mathcal{O}\big|_{U_{ij}}^k$$

where $U_{ij} = U_i \cap U_j$. Such an endomorphism can be given by a $k \times k$ matrix of regular functions on the intersections, and these matrices satisfy the conditions necessary to give a vector bundle. We go the other way, from vector bundles to sheaves, by taking the sheaf of sections of the vector bundle.

Our last important correspondence is between line bundles (a.k.a. locally free sheaves of rank one, also known as "invertible sheaves") and divisors (up to linear equivalence). This correspondence comes because we can always find local defining equations $f_i(x)$ (defined on U_i) for a divisor D and so we can make a line bundle with transition functions $a_{ij}(x) = f_i(x)/f_j(x)$ (the transition functions are 1×1 matrices). The line bundle/sheaf associated to a divisor D might be written as $\mathcal{L}(D)$ or $\mathcal{L}_D$ or $\mathcal{O}_C(D)$ or sometimes, by abuse of notation, as D. Note that $\mathcal{O}_C(D)^{-1} = \mathcal{O}_C(-D)$. Note that we are familiar with the child's first sheaf $\mathcal{O}_C$ as the line bundle $\mathcal{O}_C(0)$.

1.1. **Extensions of Line Bundles.**

We can consider $\mathcal{O}_C$-homomorphisms of our locally trivial sheaves, and these homomorphisms are just giant collections of locally defined homomorphisms of $\mathcal{O}_C$-modules. Thus we can form exact sequences and study them. In particular, we will be looking at short exact sequences of locally free sheaves. A short exact sequence of the form

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is called an extension of $\mathcal{F}''$ by $\mathcal{F}'$. For short exact sequences, ranks are additivie, i.e.

$$\mathrm{rk}\mathcal{F} = \mathrm{rk}\mathcal{F}' + \mathrm{rk}\mathcal{F}''.$$

This is completely clear on the level of modules and also holds when all of the sheaves involved are locally free (note that it is entirely possible to make an exact sequence where some of the sheaves are locally free but some have some torsion!).

So, in particular, we will be considering extensions of line bundles (so the "ends" of the SES are free of rank one and the middle is free of rank two). The first important result we have is that we always have such a sequence to look at.

Proposition 1. *A rank two bundle is always an extension of line bundles.*

This comes about essentially because a ruled surface always has a section, but you have to trust me on this because it's not a story that we have time to get into. But I will tell you why having a section makes any difference, at least in a vague way.

Take a section s of a ruled surface. We may as well think of this as the infinite section, and so it identifies infinity on each of our fibers $\mathbb{P}^1$. Once I have done this

I am able to change fiber coordinates if necessary so that my transition matrices for the ruled surface are of the form

$$\begin{bmatrix} a_{ij} & 0 \\ b_{ij} & 1 \end{bmatrix}$$

Then I see that I have a sub line bundle given by 1 (this is the trivial line bundle $\mathcal{O}_C(0)$ and a quotient line bundle a_{ij}. In other words, I have an extension of the form

$$0 \to \mathcal{O}_C \to E \to H \to 0.$$

The quotient line bundle formed in this case is called the determinant bundle (the transition functions are given by the determinant of the original transition functions). We should note that in terms of sheaves, all of this amounts to taking a given subsheaf of E and tensoring the sheaf E by its inverse so that we can assume the given subsheaf is trivial. This gives us an isomorphic subbundle. And the invariant that we will soon define is clearly unaltered by the tensoring.

We'll need the following fact about determinants, namely that for any short exact sequence of locally free sheaves

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

the determinants behave as follows

$$\det \mathcal{F} = \det \mathcal{F}' \otimes \det \mathcal{F}''.$$

Note that once again, this is clear on the local level, where we are just dealing with the determinants of matrices.

Another very nice result (but one I don't think we end up using!):

Proposition 2. *A useful result: If L_1 and L_2 are distinct subbundles of E such that*

$$\deg E = \deg L_1 + \deg L_2$$

then $E \cong L_1 \oplus L_2$.

Essentially this is done by noting that

$$\det E \otimes L_1^{-1} \otimes L_2^{-1} \cong \mathcal{O}_C,$$

which is true since we note that the degree of the former is zero and yet since L_1 and L_2 are distinct we can conclude that $H^0(C, \det E \otimes L_1^{-1} \otimes L_2^{-1}) \neq 0$. (How? We must have a nonzero homomorphism from L_2 to $\det E \otimes L_1^{-1}$ – compose the first map of the subbundle sequence for L_2 with the second map of the subbundle sequence for L_1. But this tells us that $\operatorname{Hom}(L_2, \det E \otimes L_1^{-1}) \neq 0$ but $\operatorname{Hom}(L_2, \det E \otimes L_1^{-1}) \cong H^0(C, \det E \otimes L_1^{-1} \otimes L_2^{-1})$ – here we use a little fact that $\operatorname{Hom}(A, B) \cong H^0(C, B \otimes A^{-1})$ where the sheaves are locally free rank 1 and the homomorphisms are over $\mathcal{O}_C$. You know this fact – locally it is just the linear algebra result $\operatorname{Hom}(V, W) \cong V^\vee \otimes W$.)

2. Maximal Subbundles

Proposition 3. *For a given rank two vector bundle E, the degree of a subbundle is bounded above.*

Proof. We know we always have an extension of line bundles

$$0 \to L_1 \to E \to L_2 \to 0.$$

Let L be any subbundle of E. If $L \subset L_1$ then since they have equal ranks they must be the same so that $\deg L = \deg L_1$. Suppose $L \not\subset L_1$. Then as in the previous proof we must have a nonzero homomorphism $L \to L_2$ which tells us that $H^0(C, L^{-1} \otimes L_2) \neq 0$ so that the degree of $L^{-1} \otimes L_2$ must be nonnegative, hence $\deg L_2 \geq \deg L$. Thus we have that $\deg L$ is bounded by $\max(\deg L_1, \deg L_2)$. □

Thus we can think about "maximal subbundles" – that is subbundles of maximal degree. This is more or less the same as thinking of minimal sections of ruled surfaces (classically – minimum directrix curves).

Lange and Narasimhan tackle the question of how many maximal line subbundles exist – specifically they show exactly when an old conjecture about when the subscheme of maximal line bundles is zero fails to be true. Their strategy is to translate the problem, as did Michael Atiyah in [1], into a problem about secant varieties. In doing so they find a stability criteria for extensions of line bundles in terms of secant spaces of the base curve in a particular embedding. This is what interests us.

Definition 1. . We define the s-invariant of a rank two bundle E as

$$\begin{aligned} s(E) &= \deg E - 2\max(\deg L) && \text{for } L \text{ a subbundle, or} \\ &= 2\min(\deg M) - \deg E && \text{for } M \text{ a quotient bundle.} \end{aligned}$$

E is *stable* (resp. *semistable*) if $s(E) \geq 1$ (resp. $s(E) \geq 0$).

3. Extension Spaces

Definition 2. The space $\operatorname{Ext}(L_2, L_1)$ is defined to be all of the extensions of L_2 by L_1, i.e. all the short exact sequences of $\mathcal{O}_C$-modules

$$0 \to L_1 \to E \to L_2 \to 0.$$

We define the invariant s for an extension as well – for an extension (e) as above $s(e) = s(E)$.

Proposition 4. *For L_1 and L_2 invertible sheaves and ω the canonical sheaf on C*

$$\begin{aligned} \operatorname{Ext}(L_2, L_1) &\cong \operatorname{Ext}(\mathcal{O}_C, L_1 \otimes L_2^{-1}) \\ &\cong H^1(C, L_1 \otimes L_2^{-1}) \qquad (*) \\ &\cong H^0(C, \omega \otimes L_1^{-1} \otimes L_2)^\vee \end{aligned}$$

Thus for an extension of line bundles

$$0 \to \mathcal{O}_C \to E \to H \to 0$$

we have

$$\operatorname{Ext}(H, \mathcal{O}_C) \cong H^0(C, K + H)^\vee$$

Where K is the canonical divisor and we are again abusing notation by writing the same symbol for a divisor and its associated sheaf and line bundle.

Proof. The first line of (*) comes from tensoring the original extension by L_2^{-1} (and by L_2 to go the other way). The middle line comes from a more technical definition of Ext^r as the right-derived functor of Hom – once you believe that these two definitions give the same thing for Ext^1 then you note that $\mathrm{Hom}(\mathcal{O}_C, \cdot)$ is the same functor as $\Gamma(C, \cdot)$ but then H^r is the right derived functor of Γ so all is well (this story can be found in [2, III.6]). The last line comes from Serre duality. ☐

Note that the zero element of Ext corresponds to the split exact sequence.

Now we want to consider extensions up to isomorphism, where an isomorphism of extensions is an isomorphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & L'' \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "\cong"'] & L' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L'' \arrow[r] & \tilde{E} \arrow[r] & L' \arrow[r] & 0.
\end{tikzcd}$$

In the space Ext, isomorphisms correspond to multiplication by a constant. Thus we have the space

$$\mathbb{P}\mathrm{Ext}(H, \mathcal{O}_C) \cong \mathbb{P}H^0(C, K + H).$$

3.1. **Bounds on** $s(e)$.

Suppose that we are given a non-split (i.e. nonzero in Ext) sequence

$$0 \to \mathcal{O}_C \to E \to H \to 0 \tag{e}$$

with $\deg E = \deg H = d$. Now suppose that there is a subbundle L with $\deg L \geq d$ and an associated extension

$$0 \to L \to E \to M \to 0. \tag{e'}$$

We compose the first map of (e') with the second map of (e) to get a nonzero homomorphism $L \to H$ which tells us that $\mathrm{Hom}(L, H) \cong H^0(C, H - L) \neq 0$. But

$$\deg(H - L) = \deg H - \deg L \leq 0,$$

so it must be that $H = L$. But then the original sequence is split. Thus we have an upper bound on the degree of a subbundle ($\leq d - 1$). We also have a lower bound on the maximal degree since we already know we have a subbundle of degree zero.

Thus we have the following proposition.

Proposition 5. *Let E be a fixed rank two vector bundle of degree d (that is not a direct sum $\mathcal{O}_C \oplus L$ – we can deal with those separately) with nonsplit extension*

$$0 \to \mathcal{O}_C \to E \to H \to 0.$$

Then $2 - d \leq s(E) \leq d$.

4. Curve Embedded in Extension Space

Given a base point free linear series, we can make a map from the curve to a projective space. How do we do this? Take a linear series, say a complete series associated to a divisor D. We choose a basis $f_0, \ldots, f_n$ for the vector space $L(D)$ of functions with poles no worse than $-D$ and we map

$$P \in C \mapsto (f_0(P) : \cdots : f_n(P)).$$

The fact that $|D|$ is base point free tells us that this map is well defined (not all of the f_i vanish at any one point P). We have another, coordinate-free (and hence more legitimate) version of the map, namely

$$\begin{array}{rccl} \phi_{|D|}: & C & \longrightarrow & \mathbb{P}L(D)^\vee \\ & P & \longmapsto & \{\, D' \sim D \mid D' - P \geq 0 \,\} \end{array}.$$

Now we see that given rank two vector bundle E and extension

$$\text{(e)} \qquad 0 \to \mathcal{O}_C \to E \to H \to 0$$

we can take the map $\phi_{|K+H|}$ and use it to map the curve C into the space

$$\mathbb{P}H^0(C, K+H)^\vee \cong \mathbb{P}\mathrm{Ext}(H, \mathcal{O}_C).$$

What good is this? The good is that we can tell something about the extensions by looking at the geometry of the embedded curve.

Definition 3. For $\phi : C \to \mathbb{P}^k$ an embedding, let $D = \sum n_i P_i$ be an effective divisor on C. Define the Span$D$ to be the projective linear subspace spanned by all the n_i^{th} osculating space to $\phi(C)$ of the points $\phi(P_i)$.

For our particular embedding $\phi_{|K+H|}$ and for $1 \leq j \leq d = \deg E$ we can define

$$\mathrm{Sec}_j C = \bigcup_{\deg D = j} \mathrm{Span} D.$$

In our case, we really only care about embeddings, but we can make a definition and proceed with our work even for cases when our map is not an embedding.

The n^{th} osculating space to the curve at P is the unique $\mathbb{P}^{n-1}$ in which has contact of order at least n with the curve at P. So for the easiest case, for $D = P_1 + \cdots + P_r$ with the P_i distinct, Span$D$ is the $\mathbb{P}^{r-1}$ spanned by the r points $\phi(P_i)$.

What is this span of D in our situation? To simplify things, assume D consists of distinct points. Remember that each point of our target projective space is (in affine coordinates) a hyperplane of $H^0(C, K+H)$. The ones corresponding to points $P \in C$ are hyperplanes consisting of functions of $H^0(C, K+H-P)$. When we put all of the $H^0(C, K+H-P)$ together where P is a point of the divisor A, then what we get is a function of $H^0(C, K+H-A)$. This function is also in $H^0(C, K+H)$ so it is an element of the coker$(H^0(C, K+H-A) \to H^0(C, K+H))$ and thus the hyperplane is an element of $\ker(H^0(C, K+H)^\vee \to H^0(C, K+H-A)^\vee)$. Thus we have the following.

Proposition 6. *Consider our usual setup* $\phi_{|K+H|} : C \to \mathbb{P}H^0(C, K+H)^\vee$. *For an effective divisor A,* $(e) \in \mathrm{Span}A$ *if and only if* $(e) \in \ker(H^0(C, K+H)^\vee \to H^0(C, K+H-A)^\vee$.

5. Results

Proposition 7. *Given our usual setup with fixed vector bundle E of degree d and extension*

$$(e) \qquad 0 \to \mathcal{O}_C \to E \to H \to 0,$$

suppose that M is a quotient bundle of E of minimal degree $\frac{s(E)+d}{2}$. Then $M = \mathcal{O}_C(A)$ for some effective divisor A of the same degree.

Proof. We need only show that there is an effective divisor equivalent to M. But if we consider the composition of the first map of the above sequence with the quotient map $E \to M \to 0$, this composition cannot be zero, so as we have seen before, $H^0(C, M) \neq 0$. □

Proposition 8. *For our usual setup with a fixed vector bundle E and $\phi_{|K+H|}$ embedding the curve C in the space of extensions, the extension (e) as above is in* Span$A$ *for an effective divisor A if and only if $H - A$ is a subbundle of E (i.e. if and only if A is a quotient bundle of E)*

Proof. Suppose that L is a subbundle of E; then the top of the following pull back of extensions splits (we have an inclusion $L \to E$ and so by the universal property we get a map from L to the fiber product which gives identity on L when composed with the projection from the fiber product).

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_C \arrow[r] \arrow[d, equal] & E \times_H L \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_C \arrow[r] & E \arrow[r] & H \arrow[r] & 0.
\end{tikzcd}$$

Since the top splits, we must have that (e) is in the kernel of the map of extensions of H by $\mathcal{O}$ to extensions of L by $\mathcal{O}$. In other words, $(e) \in \ker(H^0(C, K + H)^\vee \to H^0(C, K + L)^\vee)$. In the case we are concerned with, $L = H - A$, we get that $H - A$ a subbundle of E implies that $(e) \in$ Span$A$ by Proposition 6. Everything works the same going the other way.

(The final comment follows since when we consider the sequence

$$0 \to H - A \to E \to M \to 0$$

we must have that $\det E = (H - A) \otimes M$, i.e. $M \cong A$.) □

Definition 4. For an extension $(e) \in \mathbb{P}H^0(C, K + H)$ we define $h(e) = h$ if $(e) \in \mathrm{Sec}_h C - \mathrm{Sec}_{h-1} C$.

Theorem 1. *For an extension $(e) \in \mathbb{P}H^0(C, K + H)$ with E as above and $d = \deg E$,*

$$s(e) = 2h(e) - d.$$

Proof. Suppose that $(e) \notin \mathrm{Sec}_{h-1} C$ but $(e) \in \mathrm{Sec}_h C$ (so $h = h(e)$). Then we get that $(e) \in$ Span$A$ for some effective A of degree h but that there is no A of degree $h - 1$ that works. So there is a subbundle $H - A$ of degree $d - h$ but no such subbundle of higher degree. Thus $d - h$ is the maximal degree for a subbundle so that $s(e) = d - 2(d - h) = 2h - d$. □

Thus we get the final result (using Johnsen's notation [3])

Theorem 2. *Let (e) be the syndrome of a received message using $C(D, G)$. Then we can correctly decode the message if and only if (e) is unstable. In this case, to locate the error positions we must find the unique effective divisor A of degree $h(e)$ such that A is a quotient of the vector bundle E that is the middle term of (e) (since we are looking for the unique A with $(e) \in$ Span$A$).*

REFERENCES

1. M. F. Atiyah, *Complex fibre bundles and ruled surfaces*, Proc. London Math. Soc. (3) **5** (1955), 407–434. MR 17,894b
2. Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977.
3. Trygve Johnsen, *Rank two bundles on algebraic curves and decoding of Goppa codes*, preprint.
4. H. Lange and M. S. Narasimhan, *Maximal subbundles of rank two vector bundles on curves*, Math. Ann. **266** (1983), 55–72.
5. Igor R. Shafarevich, *Basic algebraic geometry 2*, second ed., Springer-Verlag, New York, 1994 (English), translated by Miles Reid.