Introduction (DISCLAIMER: Fix k, everything is over k)

Dfn An abelian variety is a complete, connected algebraic group

Dfn An algebraic group is an algebraic variety G along with regular maps $m: G \times G \to G$, $e: * \to G$, $inv: G \to G$ such that the following diagrams commute:

$$\begin{tikzcd}
* \times G \arrow[r, "e\times id"] \arrow[dr, "\sim"'] & G \times G \arrow[d, "m"] & G \times * \arrow[l, "id\times e"'] \arrow[dl, "\sim"] \\
 & G &
\end{tikzcd}$$

(Identity)

$$\begin{tikzcd}
G \arrow[r, "(inv, id)"] \arrow[d] & G \times G \arrow[d, "m"] & G \arrow[l, "(id, inv)"'] \arrow[d] \\
* \arrow[r, "e"] & G & * \arrow[l, "e"']
\end{tikzcd}$$

(Inverses)

$$\begin{tikzcd}
G \times G \times G \arrow[r, "id \times m"] \arrow[d, "m \times id"'] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"'] & G
\end{tikzcd}$$

(Associativity)

Dfn A variety X is complete if every projection map $X \times Y \to Y$ is closed.

Examples:
- Elliptic curves
- Weil restriction $Res_{K/\mathbb{Q}} E$ of an elliptic curve E
- Jacobian varieties of curves

Plan:
- Some motivation, via elliptic curves
- Gathering some material about "completeness"
- Prove that abelian varieties are abelian

Elliptic Curves ($\mathrm{char}(k) \neq 2,3$)

Thm: TFAE:

E is a projective curve over k

(a) given by $Y^2Z = X^3 + aXZ + bZ^3$, $4a^3 + 27b^2 \neq 0$

(b) which is nonsingular of genus 1 with a distinguished pt P_0.

(c) which is nonsingular with an algebraic gp structure

(d) (if $k \subseteq \mathbb{C}$) such that $E(\mathbb{C}) = \mathbb{C}/\Lambda$ for some lattice $\Lambda \subseteq \mathbb{C}$.

Dfn An elliptic curve over k is any/all of that.

Proof: Strategy $(a) \Leftrightarrow (b) \Leftrightarrow (c)$, $(b) \Rightarrow (d) \Rightarrow (a)$

$((a) \Rightarrow (b))$: ✓

$((b) \Rightarrow (a))$: Riemann-Roch states $\ell(D) = \ell(K-D) + \deg(D) + 1 - g$

So here, $\ell(D) = \ell(K-D) + \deg(D)$

Further, if $D > 0$, $\ell(K-D) = 0$, in which case $\ell(D) = \deg(D)$

Consider $L(nP_0)$, $n>0$.

$RR \Rightarrow \ell(nP_0) = n$

Then always contain the constants.

$$L(P_0) = k \qquad \ldots \quad L(6P_0) = \begin{pmatrix} k \oplus kx \oplus ky \oplus kx^2 \\ \oplus ky^2 \oplus kxy \oplus kx^3 \end{pmatrix} /\sim$$

$$L(2P_0) = k \oplus kx$$

$$L(3P_0) = k \oplus kx \oplus ky$$

So we must have a relation, which after manipulation is of the desired form.

We get an embedding

$$\begin{aligned} E &\hookrightarrow \mathbb{P}^2 \\ P &\longmapsto (x(P) : y(P) : 1) \qquad (P \neq P_0) \\ P_0 &\longmapsto (0 : 1 : 0) \end{aligned}$$

and thus E is of the desired form.

$((b) \Rightarrow (c))$: Consider $D \in \mathrm{Pic}^0(E)$

$RR \Rightarrow \ell(D + (P_0)) = 1$

So $L(D+(P_0)) = k \cdot f$, for some $f \in k(E)$.

Since $\deg((f)) = 0$,

$(f) = -D - (P_0) + (P_D)$, for some $P_D \in E$.

(This is in fact unique)

So we have a bijection

$$\begin{aligned} \mathrm{Pic}^0(E) &\longrightarrow E \\ D &\longmapsto P_D \\ 0 &\longmapsto P_0 \end{aligned}$$

From this we define a group law on E.

((c) $\Rightarrow$ (b)): Group structure $\Rightarrow$ Canonical sheaf ω (cotangent bundle) is trivial

K is the divisor corresponding to ω, so $\deg(K) = 0$.

RR with $D = 0$ $\Rightarrow$ $\ell(K) = g$

RR with $D = K$ $\Rightarrow$ $\deg(K) = 2g-2$.

Thus $2g-2 = 0$, $\Rightarrow$ $g = 1$.

((b) $\Rightarrow$ (d)): $E(\mathbb{C})$ is a Riemann surface of genus 1, ie. a complex 1-torus.

$\Rightarrow$ $E(\mathbb{C}) \cong \mathbb{C}/\Lambda$

((d) $\Rightarrow$ (a)): The Weierstrass $\wp$-function of Λ and its derivative satisfy

$$[\wp'(z)]^2 = 4\wp(z)^3 + g_2\wp(z) + g_3 \qquad g_2, g_3 \in \mathbb{C}.$$

So we take the embedding

$$\begin{aligned} \mathbb{C}/\Lambda &\longrightarrow \mathbb{P}^2 \\ z &\longmapsto (\wp(z) : \wp'(z) : 1) \qquad z \neq 0 \\ 0 &\longmapsto (0:1:0) \end{aligned}$$

$\square$

Generalising to abelian varieties?

(a): No, in general we don't know what the equations look like

(b): One could potentially replace "genus" with a condition on the dimension of cohomology groups

(c): Yes, this is essentially our definition!

(d): Yes, stay tuned!

Complete Varieties

Idea: If $X \times Y$ had product topology (instead of its Zariski topology), then complete $\Leftrightarrow$ compact.

We'd like to gather a few results about complete varieties we can use to access properties of abelian varieties (like abelianness).

Prop: Let V be a complete variety.
Given any morphism $\varphi : V \to W$, $\varphi(V)$ is closed.

Pf: Let $\Gamma_\varphi = \{(v, \varphi(v))\} \subseteq V \times W$ be the graph of W.
It's a closed subvariety of $V \times W$.
Under the projection $V \times W \to W$, the image of Γ_φ is $\varphi(V)$, and thus closed. $\square$

Cor: If V is complete and connected, any regular function on V is const.

Pf: A regular function is a morphism $f : V \to \mathbb{A}^1$.
By the above, $f(V) \subseteq \mathbb{A}^1$ is closed, and thus a finite set of points.
Connected $\Rightarrow$ Just one point. $\square$

Cor: Let V be a complete, connected variety.
Let W be an affine variety.
Given $\varphi: V \to W$, then $\varphi(V)$ is a point.

Pf: We have an embedding $W \hookrightarrow \mathbb{A}^n$.
On $\mathbb{A}^n$ we have the coordinate functions $\mathbb{A}^n \xrightarrow{x_i} \mathbb{A}^1$.
The composition

$$V \xrightarrow{\varphi} W \hookrightarrow \mathbb{A}^n \to \mathbb{A}^1$$

by the above is constant.
Thus the coordinates of $\varphi(V)$ are constant, so $\varphi(V) = \{pt\}$. $\square$

A final side result of interest that I won't prove today:

Thm: Projective varieties are complete.

The main goal of this section is to prove the following theorem.

Thm (Rigidity Thm)

Let V, W be varieties such that V is complete and $V \times W$ is geometrically irreducible.
Let $\alpha: V \times W \to U$ be a morphism such that $\exists\, u_0 \in U(k)$, $v_0 \in V(k)$, $w_0 \in W(k)$ with

$$\alpha(V \times \{w_0\}) = \alpha(\{v_0\} \times W) = \{u_0\}.$$

Then $\alpha(V \times W) = \{u_0\}$

Pf: Since $V \times W$ is geometrically irreducible, V must be connected.
Denote the projection $q: V \times W \longrightarrow W$.
Let $U_0 \ni u_0$ be an open neighbourhood.
We consider the set

$$Z = \{ w \in W \mid \alpha((v,w)) \notin U_0 \text{ for some } v \in V \}$$
$$= q(\alpha^{-1}(U \setminus U_0))$$

Since q is closed, $Z \subseteq W$ is closed.
Since $w_0 \in W \setminus Z$, $W \setminus Z$ is a nonempty open subset of W.

Consider $w \in W \setminus Z$.
Since $V \times \{w\} \cong V$, it is complete and connected.
Thus

$$\alpha(V \times \{w\}) = \{pt\} = \alpha((v_0, w)) = \{u_0\}.$$

$$\Rightarrow \alpha(V \times (W \setminus Z)) = \{u_0\}$$

Since $V \times (W \setminus Z) \subseteq V \times W$ is open and $V \times W$ is irreducible, it is dense.

$$\Rightarrow \alpha(V \times W) = \{u_0\}.$$

$\square$

Abelian Varieties are Abelian

Prop: Let A, B be abelian varieties.
Every morphism $\alpha : A \to B$ is the composition of a homomorphism and a translation.

Pf: First, compose by a translation on B such that $\alpha(0) = 0$.
Consider the map

$$\varphi : A \times A \longrightarrow B$$
$$(a, a') \mapsto \alpha(a+a') - \alpha(a) - \alpha(a').$$

Then

$$\varphi(A \times \{0\}) = \alpha(a+0) - \alpha(a) - \alpha(0) = 0$$
$$\varphi(\{0\} \times A) = \alpha(0+a') - \alpha(0) - \alpha(a') = 0$$

By the Rigidity theorem, $\varphi(A \times A) = \{0\}$.

$$\Rightarrow \alpha(a+a') = \alpha(a) + \alpha(a').$$ $\square$

Cor: Abelian varieties are abelian.

Pf: The inversion map $a \mapsto -a$ sends 0 to 0 and is thus a homomorphism.
Thus $a+b-a-b = a+b-(a+b) = 0$

$$\Rightarrow a+b = b+a$$ $\square$