BUNTES

Alex Best

Nov 10, 2017

Rosati Involution

Let A/k, consider $f \in \mathrm{End}(A)$.

We can associate

$$f \longmapsto \hat{f} \in \mathrm{End}(\hat{A}).$$

Given a polarisation

$$\lambda : A \longrightarrow \hat{A}$$

we can instead associate

$$f \longmapsto \lambda \circ f \circ \lambda^{-1} \in \mathrm{End}^0(\hat{A}) = \mathrm{End}(A) \otimes \mathbb{Q}$$

When do these coincide?

That is, when is

$$f = \lambda^{-1} \circ \hat{f} \circ \lambda =: f^{\dagger} \ ?$$

Dfn: The Rosati involution is

$$\cdot^{\dagger} : \mathrm{End}^0 A \longrightarrow \mathrm{End}^0 A$$
$$f \longmapsto \lambda^{-1} \circ \hat{f} \circ \lambda$$

Properties:

1. $\cdot^{\dagger}$ is $\mathbb{Q}$-linear
2. $(fg)^{\dagger} = g^{\dagger} f^{\dagger}$, ie. $\cdot^{\dagger}$ is an antihomomorphism
3. Recall the ℓ-adic Weil pairing for $\ell \neq \mathrm{char}(k)$, let $a, a' \in V_\ell A = T_\ell A \otimes \mathbb{Q}$

   Then

$$e_\ell^{\lambda}(fa, a') = e_\ell^{\lambda}(a, f^{\dagger} a').$$

Pf of ③: $e_\ell^\lambda(fa, a') = e_\ell(fa, \lambda a')$
$= e_\ell(a, \hat{f}\lambda a')$
$= e_\ell(a, \lambda \cdot \lambda^{-1} \cdot \hat{f} \cdot \lambda \cdot a')$
$= e_\ell(a, \lambda \cdot f^\dagger a')$
$= e_\ell^\lambda(a, f^\dagger a')$ □

④ The Rosati involution is an involution (ie. $f^{\dagger\dagger} = f$).

Pf of ④: By ③, $e_\ell^\lambda(fa, a') = e_\ell^\lambda(a, f^\dagger a')$
$= -e_\ell^\lambda(f^\dagger a', a)$ (e_ℓ^λ is antisymmetric)
$= -e_\ell^\lambda(a', f^{\dagger\dagger} a)$
$= e_\ell^\lambda(f^{\dagger\dagger} a, a')$

Since e_ℓ^λ is nondegenerate, $f = f^{\dagger\dagger}$. □

How can we understand this further?

$$\mathrm{End}^\circ(A) \times \mathrm{End}^\circ(A) \longrightarrow \mathbb{Q}$$
$$(f, g) \longmapsto \mathrm{Tr}(fg^\dagger)$$

Proposition: This is positive definite.
In particular,

$$\mathrm{Tr}(ff^\dagger) = 2g \frac{(D^{g-1} \cdot f^\dagger D)}{(D^g)} \leftarrow \text{intersection number}$$

where $\lambda = \lambda_{\mathcal{L}(D)}$.

For a simple abelian variety, $\mathrm{End}^\circ(A)$ is a finite division algebra $/\mathbb{Q}$.

Thus we have a finite division algebra $/\mathbb{Q}$ with a positive definite involution.

Call such things <u>Albert algebras</u>.

<u>Theorem</u> [Albert, 1934/5]

Let $(D, ')$ be an Albert algebra with involution $'$.
Then let K be the centre of D, and K_0 be the subfield fixed by $'$.
Then $(D, ')$ is one of 4 types:

(I) $D = K = K_0$ is a totally real number field, and $'$ is trivial.

(II) D is a quaternion algebra $/K = K_0$ a totally real field, that is split at all infinite places.

For a quaternion algebra we have a standard conjugation defined by

$$x + x^* = \mathrm{Tr}(x).$$

Then let

$$x' = a x^* a^{-1}$$

for some $a \in D$ such that $a^2 \in K$ is totally negative.

(III) D is a quaternion algebra $/K = K_0$ a totally real number field, that is ramified at all infinite places.

Then $x' = x^*$, the standard conjugation on D.

(IV) D is a division algebra $/K$ a CM field (ie a totally imaginary quadratic extension of a totally real number field), and K_0 is the maximal totally real subfield. + further conditions on D...

Then $x' = a x^* a^{-1}$ as in (II).

Let $g = \dim(A)$

D
$| \, d^2$
K
$|$ (total degree e)
K_0
$| \, e_0$
$\mathbb{Q}$

type	e	d	in char 0	in char p
I	e_0	1	$e \mid g$	$e \mid g$
II	e_0	2	$2e \mid g$	$2e \mid g$
III	e_0	2	$2e \mid g$	$e \mid g$
IV	$2e_0$	d	$e_0 d^2 \mid g$	$e_0 d \mid g$

Let's come back to our earlier question, what is

$$S = \{ f \in \mathrm{End}^0(A) \mid f^\dagger = f \} \ ?$$

We have a map

$$\mathbb{Q} \otimes NS(A) \longrightarrow \mathrm{Hom}^0(A, \hat{A}) \xrightarrow{\sim} \mathrm{End}^0(A)$$
$$= \mathbb{Q} \otimes \frac{\mathrm{Pic}\, A}{\mathrm{Pic}^0 A}$$

$$\mathcal{M} \longmapsto \lambda_{\mathcal{M}}$$

$$\phi \longmapsto \lambda^{-1} \phi$$

(λ = some fixed polarisation)

Proposition: Let $k = \bar{k}$.

Then the image of the above composition is S.

Proof: Fix $\alpha \in \mathrm{End}^0(A)$, $\ell \neq \mathrm{char}(k)$, odd.

Apply 13.6, see $\lambda\alpha = \phi_\mathcal{L} \iff e_\ell^{\lambda\alpha}$ is skew-symmetric.

$$e_\ell^{\lambda\alpha}(a, a') = e_\ell^{\lambda}(a, \alpha a') = -e_\ell^{\lambda}(\alpha a', a) = -e_\ell^{\lambda}(a', \alpha^{\dagger} a)$$
$$= -e_\ell(a', \hat{\alpha}\lambda a)$$

However, $e_\ell^{\lambda\alpha}$ skew-symmetric $\iff e_\ell^{\lambda\alpha}(a, a') = -e_\ell^{\lambda\alpha}(a', a)$
$$= -e_\ell(a', \lambda\alpha a)$$

Non-degeneracy of $e_\ell \Rightarrow \hat{\alpha}\lambda = \lambda\alpha \iff \alpha^{\dagger} = \alpha$. $\square$

One last cool result we can now prove.

Theorem: If (A, λ) is a polarized abelian variety, then $\mathrm{Aut}(A, \lambda)$ is finite.

Proof: Assume $\alpha \in \mathrm{End}(A)$ such that $\lambda\alpha = \hat{\alpha}\lambda$.

Then $\alpha^{\dagger}\alpha = 1 \Rightarrow \mathrm{Tr}(\alpha^{\dagger}\alpha) = 2g$.

So
$$\alpha \in \mathrm{End}(A) \cap \{\beta \in \mathrm{End}^0(A) \mid \mathrm{Tr}(\beta^{\dagger}\beta) = 2g\}$$

However, $\mathrm{End}(A)$ is discrete in $\mathrm{End}^0(A)$, but the second factor above is compact. $\square$