BUNTES

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Abelian Varieties over $\mathbb{C}$

Goal: Study these

Alex's goal: What is a polarisation?

Recall: For E an elliptic curve $/\mathbb{C}$

$$\begin{aligned} \mathbb{C}/\Lambda &\xrightarrow{\sim} E(\mathbb{C}) \subseteq \mathbb{P}^2(\mathbb{C}) \\ z &\longmapsto (\wp(z) : \wp'(z) : 1) \\ 0 &\longmapsto (0 : 1 : 0) \end{aligned}$$

where

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right).$$

This is a meromorphic function who's image lands in

$$y^2 = 4x^2 - g_2 x - g_3.$$

Let A be an abelian variety over $\mathbb{C}$.

What does $A(\mathbb{C})$ look like?

Proposition

$A(\mathbb{C})$ is a compact, connected, complex Lie group.

Proposition

$A(\mathbb{C}) \cong V/\Lambda$, where V is a g-dimensional $\mathbb{C}$-vector space, and Λ is a full rank lattice $\Lambda \subseteq V$.

Note: · Full rank means $\mathbb{R} \otimes \Lambda = V$
· Lattice means discrete subgroup of V

Proof: Differential geometry $\Rightarrow$ $\exp : \mathrm{Tgt}_0 A(\mathbb{C}) \longrightarrow A(\mathbb{C})$.

This is holomorphic.
Since $A(\mathbb{C})$ is abelian, this is a homomorphism.
Locally an isomorphism at 0.

Claim: exp is surjective.

There exists a neighbourhood $U \ni 0$ s.t. $\exp(U) \cong U$.
Consider $\exp(\mathrm{Tgt}_0 A(\mathbb{C}))$.
For $x \in \exp(\mathrm{Tgt}_0 A(\mathbb{C}))$, $\{U + x\}$ are all open an give a cover. Thus $\exp(\mathrm{Tgt}_0 A(\mathbb{C}))$ is open.

Since $A(\mathbb{C})$ is connected, we are thus reduced to showing
$\exp(\mathrm{Tgt}_0 A(\mathbb{C}))$ is closed also.

Since exp is a homomorphism, the image is a subgroup.
So its complement is the union of its cosets, which is open.
Thus $\exp(\mathrm{Tgt}_0 A(\mathbb{C}))$ is closed.

$\Rightarrow \exp(\mathrm{Tgt}_0 A(\mathbb{C})) = A(\mathbb{C})$

Thus the claim.

exp is a local isomorphism
$\Rightarrow \ker(\exp)$ is discrete
$\Rightarrow \ker(\exp)$ is a lattice.

We now have $A(\mathbb{C}) \cong \mathrm{Tgt}_0 A(\mathbb{C}) / \ker(\exp)$.

$A(\mathbb{C})$ is compact $\Rightarrow \mathrm{Tgt}_0 A(\mathbb{C}) / \ker(\exp)$
$\Rightarrow \ker(\exp)$ is full rank. $\square$

From this isomorphism we can now read off properties of $A(\mathbb{C})$ as a group.

We call V/Λ a <u>complex torus</u>.

Proposition

$A(\mathbb{C})$ is divisible, and $A(\mathbb{C})[n] = (\mathbb{Z}/n\mathbb{Z})^{2g}$.

Proof: $A(\mathbb{C}) \cong V/\Lambda \underbrace{\cong}_{\text{as groups}} (\mathbb{R}/\mathbb{Z})^{2g}$.

Thus divisible.

Further, $(\mathbb{R}/\mathbb{Z})[n] = (\frac{1}{n}\mathbb{Z})/\mathbb{Z}$.

Question: Given a complex torus V/Λ, does there exist an abelian variety A such that $A(\mathbb{C}) \cong V/\Lambda$?

Some cases:

- $\mathbb{C}/\Lambda \cong E(\mathbb{C})$ always
- $\mathbb{C}^2/\Lambda^2 \cong (E \times E)(\mathbb{C})$ sometimes yes
- $\mathbb{C}^2/(\mathbb{Z}\text{-span}\{(i,0),(i\sqrt{p}, i),(1,0),(0,1)\})$ p prime (Perhaps no? Not sure)

Theorem (Chow)

If X is an analytic submanifold of $\mathbb{P}^n(\mathbb{C})$, then X is an algebraic subvariety.

By this theorem, it is enough to analytically embed $V/\Lambda \hookrightarrow \mathbb{P}^n$.

Mimic the elliptic curve strategy: find enough functions $\Theta : V/\Lambda \longrightarrow \mathbb{C}$.

Proposition

Let $X = V/\Lambda$. Then

$$H^r(X; \mathbb{Z}) \cong \{\text{alternating } r\text{-forms } \underbrace{\Lambda \times \cdots \times \Lambda}_{r} \longrightarrow \mathbb{Z}\}$$

Proof: $\pi : V \longrightarrow V/\Lambda$ is a universal covering map.

Then

$$\Lambda = \pi^{-1}(0) \cong \pi_1(X, 0)$$

Because all these spaces are nice,

$$H^1(X, \mathbb{Z}) \cong \mathrm{Hom}(\pi_1(X), \mathbb{Z}) \cong \mathrm{Hom}(\Lambda, \mathbb{Z}).$$

To extend to $r \neq 1$, use the Künneth formula.

$$\begin{array}{ccc}
\Lambda^r(H^1(X_1 \times X_2, \mathbb{Z})) & = & H^r(X_1 \times X_2, \mathbb{Z}) \\
\| \text{ Künneth} & & \\
\Lambda^r(H^1(X_1, \mathbb{Z}) \oplus H^1(X_2, \mathbb{Z})) & & \| \text{ Künneth} \\
\| & & \\
\bigoplus_{p+q=r} \left(\Lambda^p(H^1(X_1,\mathbb{Z})) \otimes \Lambda^q(H^1(X_2,\mathbb{Z}))\right) & = & \bigoplus_{p+q=r} \left(H^p(X_1,\mathbb{Z}) \otimes H^q(X_2,\mathbb{Z})\right)
\end{array}$$

Since we know the proposition for $S^1 = \mathbb{R}/\mathbb{Z}$, by taking products and applying the above we get it for all tori V/Λ. $\square$

<u>Proposition</u>

There is a correspondence

$$\begin{array}{ccc}
\left\{ \begin{array}{c} \text{Hermitian forms} \\ H \text{ on } V \end{array} \right\} & \longleftrightarrow & \left\{ \begin{array}{c} \text{Alternating real-valued forms} \\ E : V \times V \longrightarrow \mathbb{R} \end{array} \right\} \\
H & \longmapsto & \operatorname{im} H \\
E(iu,v) + iE(u,v) & \longleftarrow & E
\end{array}$$

We now want to consider line bundles on $X = V/\Lambda$.
That is

$$\begin{array}{c} L \\ \pi \downarrow \\ X \end{array}$$ such that for any $x \in X$ there exists $U \ni x$ such that $\pi^{-1}(U) \cong \mathbb{C} \times U$.

Definition

If H is a Hermitian form on V such that $E(\Lambda \times \Lambda) \subseteq \mathbb{Z}$, there exists a map $\alpha : \Lambda \longrightarrow \mathbb{C}^* := \{ z \in \mathbb{C}^* \mid |z| = 1 \}$ such that
$\alpha(u+v) = e^{i\pi E(u,v)} \alpha(u)\alpha(v)$.
Further, there is a line bundle $L(H, \alpha)$ on X which is defined by quotienting $\mathbb{C} \times V$ by Λ, which acts by

$$\phi_u(\lambda, v) = \Big(\underbrace{\alpha(u) e^{\pi H(v,u) + \frac{1}{2}\pi (H(u,u))}}_{=: e_u} \lambda, \ v+u \Big) \quad \text{for } u \in \Lambda.$$

Theorem (Appell-Humbert)

Any line bundle on X is $L(H, \alpha)$ for some H, α as above.
Further,

$$L(H_1, \alpha_1) \otimes L(H_2, \alpha_2) = L(H_1 + H_2, \alpha_1 \alpha_2).$$

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(\Lambda, \mathbb{C}) \arrow[r] \arrow[d, "\wr"] & \{ \text{data } (H, \alpha) \} \arrow[r] \arrow[d, "\wr"] & \left\{ \begin{array}{c} \text{group of Hermitian } H \\ \text{with } E(\Lambda \times \Lambda) \subseteq \mathbb{Z} \end{array} \right\} \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & \mathrm{Pic}^0(X) \arrow[r] & \mathrm{Pic}(X) \arrow[r, "c"] & \ker\big(H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X) \big) \arrow[r] & 0
\end{tikzcd}$$

$\mathrm{Pic}^0(X)$: topologically trivial line bundles on X

$\mathrm{Pic}(X)$: line bundles on X

We wanted functions $X \longrightarrow \mathbb{C}$.

Now we can consider sections

$$\begin{tikzcd} L(H,\alpha) \arrow[d, "\pi"] \\ X \arrow[u, bend left, "s"] \end{tikzcd} \quad \text{with} \quad \pi \circ s = id.$$

Denote the space of such things as $H^0(X, L(H,\alpha))$

These correspond to holomorphic functions

$$\theta : V \longrightarrow \mathbb{C} \qquad \text{such that} \quad \theta(z+u) = \underbrace{e_u}_{\text{factor from earlier}} \theta(z)$$

Such θ is called a theta function for (H, α).

If H is not positive-definite, the space of such functions is 0.

<u>Proposition</u>

If H is positive definite, then the dimension of $H^0(\lambda, L(H,\alpha))$ is $\sqrt{\det E}$ where we write a matrix for E with respect to an integral basis.

<u>Theorem</u> (Lefschetz)

Given a positive definite H, there exists an imbedding $X \hookrightarrow \mathbb{P}^m$.

Proof (sketch): Let $L = L(H, \alpha)$.

Consider $L(H,\alpha)^{\otimes 3} = L(3H, \alpha^3)$

Take a basis $\theta_0, \dots, \theta_d$ of $H^0(X, L^{\otimes 3})$.

Claim: $\Theta: z \longmapsto (\theta_0(z) : \dots : \theta_d(z)) \in \mathbb{P}^d$ is an imbedding.

To see that this is well-defined, we must give a section of $L^{\otimes 3}$ not vanishing at z for all $z \in X$.

Let $\theta \in H^0(X, L) \setminus \{0\}$.

Then pick a, b such that the section of $L^{\otimes 3}$ given by

$$\theta(z-a)\,\theta(z-b)\,\theta(z+a+b)$$

does not vanish.

This is possible, and thus we have a nonvanishing section.

For injectivity, show that if the above section has the same value on some z_1, z_2, then it's a theta function for some sublattice.

Almost all sections aren't theta functions for a sublattice (uses previous prop about dimensions).

Similarly for tangent vectors. □

Definition

A Riemann form is $E : \Lambda \times \Lambda \to \mathbb{Z}$ alternating such that $E_{\mathbb{R}} : V \times V \to \mathbb{R}$ has the property that $E(iv, iw) = E(v, w)$ and the corresponding Hermitian form was positive definite.

Definition

A complex torus $X = V/\Lambda$ is *polarisable* if there exists a Riemann form E on Λ.

Example (Proposition)

Every $\mathbb{C}/\Lambda$, where $\Lambda = \mathbb{Z}\text{-span}\{1, \tau\}$, is polarisable.

To see this, $E(z,w) = \dfrac{z\bar{w}}{\operatorname{Im}\tau}$ is a Riemann form.

Putting everything together, we have an equivalence of categories

$$\{\text{Abelian varieties over } \mathbb{C}\} \longleftrightarrow \{\text{Polarisable complex tori}\}$$

Definition

An *isogeny* of complex tori is a homomorphism $V/\Lambda \longrightarrow V'/\Lambda'$ with finite kernel.

Definition

Given V a complex vector space, let

$$V^* = \{ f: V \to \mathbb{C} \mid f(u+v) = f(u) + f(v),\ f(\alpha v) = \bar{\alpha} f(v) \}$$

Given $\Lambda \subseteq V$ a lattice, let

$$\Lambda^* = \{ f \in V^* \mid f(\lambda) \in \mathbb{Z},\ \lambda \in \Lambda \}.$$

Definition

If $X = V/\Lambda$, $X^\vee = V^*/\Lambda^*$ is the dual torus.

Proposition

$$X \times X^\vee \longrightarrow \mathbb{C}$$

So

$$X[n] \times X^\vee[n] \longrightarrow (\tfrac{1}{n^2}\mathbb{Z})/(\tfrac{1}{n}\mathbb{Z}) \simeq (\mathbb{Z}/n\mathbb{Z}).$$

This is called the Weil pairing.

Can a torus be isogenous to its own dual?

If X is polarisable, then

$$\begin{aligned} X &\longrightarrow X^\vee \\ v &\longmapsto H(v, -) \end{aligned}$$

is an isogeny.

Definition

A polarisation is an isogeny $X \longrightarrow X^\vee$.