BUNTES

María Ines

Sep 22, 2017

§3. Rational Maps into Abelian Varieties

Note: Today, all varieties are irreducible.

Rational Maps

Let V, W be varieties $/K$.

Consider pairs (U, φ_U), where $\emptyset \neq U \subseteq V$ open subset (so dense), and $\varphi_U : U \longrightarrow W$ is a regular map.

Dfn: • Two pairs (U, φ_U), $(U', \varphi_{U'})$ are equivalent if φ_U and $\varphi_{U'}$ agree on $U \cap U'$.

• An equivalence class φ of $\{(U, \varphi_U)\}$ is a rational map $\varphi : V \dashrightarrow W$.

• $\varphi : V \dashrightarrow W$ is defined at $v \in V$ if $v \in U$ for some $(U, \varphi_U) \in \varphi$.

Note: The set $U_1 = \bigcup_{(U, \varphi_U) \in \varphi} U$, where φ is defined is open and $(U_1, \varphi_1) \in \varphi$ where $\varphi_1 : U_1 \longrightarrow W$ restricts to φ_U on U.

Example (where $U_1 \neq V$)

① Let $\emptyset \neq W \subsetneq V$ be open. Then the rational map $V \dashrightarrow W$ induced by $id : W \longrightarrow W$ will not extend to V.

To avoid this, assume W is complete.

(2) Consider $C: y^2 = x^3$

We have a regular map $\alpha: \mathbb{A}^1 \longrightarrow C$

$$\alpha \longmapsto (\alpha^2, \alpha^3)$$

which restricts to an isomorphism $\mathbb{A}^1 \setminus \{0\} \longrightarrow C \setminus \{0\}$.

The inverse of $\alpha|_{\mathbb{A}^1 \setminus \{0\}}$ represents $\beta: C \dashrightarrow \mathbb{A}^1$ which does not extend to C.

β corresponds to a map on function fields

$$K(t) \longrightarrow K(x,y)$$

$$t \longmapsto y/x$$

which does not send $K[t]_{(t)}$ to $K[x,y]_{(x,y)}$

(3) Let V be a nonsingular surface, $P \in V$.

Then $\exists\, \alpha: W \longrightarrow V$ regular that induces an isomorphism

$\alpha: W \setminus \alpha^{-1}(P) \longrightarrow V \setminus P$, but $\alpha^{-1}(P)$ is a full projective line.

The rational map represented by α^{-1} is not regular on V (i.e. where to send P?).

Theorem 3.1

A rational map $\varphi: V \dashrightarrow W$ from a nonsingular variety V to a complete variety W is defined on an open subset $U \subseteq V$ whose complement has codimension ≥ 2.

__Proof:__ (We'll consider the case that V is a curve)

Let V be a nonsingular curve, $\phi \neq U \subseteq V$ open, $\varphi : U \to V$ a regular map.

$$
\begin{tikzcd}
 & & & V & v \\
 & & & V \times W \arrow[u, "p"'] \arrow[d, "q"] & (v,w) \arrow[u, mapsto] \arrow[d, mapsto] \\
U \arrow[r, "{(u,\varphi(u))}"] \arrow[urrr, bend left] \arrow[drrr, bend right] & U' \subseteq & Z \subseteq \arrow[uur] & & \\
 & & & W & w
\end{tikzcd}
$$

U' is the image of U.

$Z = \overline{U'}$ is the closure of U'.

W complete, Z closed $\Rightarrow$ $p(Z) \subseteq V$ is closed

Also $U \subseteq p(Z)$ $\Rightarrow$ $p(Z) = V$ since U is dense.

We have $U \xrightarrow{\sim} U' \longrightarrow U$, so

$$
\begin{tikzcd}
U \arrow[r, "\sim"] \arrow[rr, bend right, "\mathrm{id}"'] & U' \arrow[r] & U
\end{tikzcd}
\qquad
\begin{tikzcd}
U' \arrow[r, "\sim"] \arrow[d, phantom, "\cap" rotate=90] & U \arrow[d, phantom, "\cap" rotate=90] \\
Z \arrow[r, two heads] & V
\end{tikzcd}
$$

Since curves are specified by their function fields, this implies $Z \xrightarrow{\sim} V$.

Then $q|_Z : Z (\simeq V) \longrightarrow W$ is the extension of φ to V.

$\square$

Rational maps into AVs

Theorem 3.2

A rational map $\alpha: V \dashrightarrow A$ from a nonsingular variety to an abelian variety must extend to all of V.

To prove this, we need the following lemma.

Lemma 3.3: Let $\varphi: V \dashrightarrow G$ from a nonsingular variety to a group variety. Then either φ is defined on all of V or the subset where φ is not defined is closed of pure codimension 1.

Proof: Fix $(\mathcal{U}, \varphi_{\mathcal{U}}) \in \varphi$ and consider $\Phi: V \times V \to G$ represented by

$$\begin{array}{ccccccc} \mathcal{U} \times \mathcal{U} & \xrightarrow{\varphi_u \times \varphi_u} & G \times G & \xrightarrow{id \times inv} & G \times G & \xrightarrow{m} & G \\ (x,y) & & \longmapsto & & & & \varphi_u(x)\varphi_u(y)^{-1} \end{array}$$

Check: φ is defined at $x \iff \Phi$ is defined at (x,x)

(In this case, $\Phi(x,x) = e$).

$\iff$ The map $\Phi^*: \mathcal{O}_{G,e} \to K(V \times V)$ induced by Φ satisfies $\mathrm{Im}(\mathcal{O}_{G,e}) \subseteq \mathcal{O}_{V\times V, (x,x)}$

For a nonzero function f on $V \times V$, write

$$\operatorname{div}(f) = \underbrace{\operatorname{div}(f)_0 - \operatorname{div}(f)_\infty}_{\text{effective divisors}}$$

Then

$$\mathcal{O}_{V\times V,(x,x)} = \{0\} \cup \{ f \in K(V \times V) \mid \operatorname{div}(f)_\infty \text{ does not contain } (x,x) \}$$

Suppose φ is not defined at x.
Then there exists $f \in \operatorname{Im}(\mathcal{O}_{a,e})$ such that $(x,x) \in \operatorname{div}(f)_\infty$.
Then Φ is not defined at any $(y,y) \in \Delta \cap \underbrace{\operatorname{div}(f)_\infty}_{= \operatorname{div}(f')_\infty}$, which is a pure codimension 1 subset of Δ by [AG, Thm 9.2]
The corresponding subset in V is of pure codimension 1, and φ is not defined there.

□

Proof of Thm 3.2:

If the rational map $\alpha : V \dashrightarrow A$ were not defined on all of V, the subset where it is not defined, by Lemma 3.3, would be of pure codim 1.
However, by Thm 3.1 this cannot occur.

Thus α must be defined on all of V.

□

Theorem 3.4

Let $\alpha : V \times W \to A$ be a morphism from a product of nonsingular varieties into an abelian variety. Assume $V \times W$ irreducible.

If
$$\alpha(V \times \{w_0\}) = \{a_0\} = \alpha(\{v_0\} \times W)$$
for some $a_0 \in A$, $v_0 \in V$, $w_0 \in W$, then $\alpha(V \times W) = \{a_0\}$.

<u>Note</u>: If V or W is complete, this is a special case of the <u>Rigidity Theorem</u> (see Lecture 1).

<u>Corollary</u>

Every rational map $\alpha : G \dashrightarrow A$ from a group variety into an abelian variety is the composition of a homomorphism $G \to A$ and a translation in A.

<u>Proof</u>: Since group varieties are nonsingular, α is a regular map by Theorem 3.2.

From here the proof is identical to that for the analogous statement in Lecture 1.

$\square$

Abelian Varieties up to Birational Equivalence

Dfn: $\varphi : V \dashrightarrow W$ is *dominating* if $\mathrm{Im}(\varphi_u)$ is dense in W for a representative $(\mathcal{U}, \varphi_u) \in \varphi$.

Exercise: A dominating $\varphi : V \dashrightarrow W$ defines a homomorphism $K(W) \to K(V)$, and any such homomorphism arises from a unique dominating rational map.

Dfn: $\varphi : V \dashrightarrow W$ is *birational* if the corresponding homomorphism $K(W) \to K(V)$ is an isomorphism.

Or, equivalently, if there exists $\psi : W \dashrightarrow V$ such that $\varphi \circ \psi$ and $\psi \circ \varphi$ are the identity where they are defined.

In this case, we say V and W are *birationally equivalent*.

Note: In general, birational equivalence $\not\Rightarrow$ isomorphic

Example: (From earlier)

① V variety, $\emptyset \neq W \subsetneq V$ an open subset.

② $V = \mathbb{A}^1$, $W : Y^2 = X^3$

Theorem 3.8

If two abelian varieties are birationally equivalent, then they are isomorphic as abelian varieties.

Proof: A, B abelian varieties.

$\varphi : A \dashrightarrow B$ a birational map with inverse ψ.

Then, by Theorem 3.2, these both extend to regular maps.

$$\varphi : A \to B, \quad \psi : B \to A$$

and $\varphi \circ \psi$, $\psi \circ \varphi$ are the identity everywhere.

$\Rightarrow \varphi$ is an isomorphism of algebraic varieties, and after composing with a translation, φ is also a group isomorphism.

$\square$

Proposition 3.9: Any rational map $\mathbb{A}^1 \dashrightarrow A$ or $\mathbb{P}^1 \dashrightarrow A$, for A an abelian variety, is constant.

Proof: Theorem 3.2 $\Rightarrow$ any $\alpha : \mathbb{A}^1 \dashrightarrow A$ extends to a regular map $\alpha : \mathbb{A}^1 \to A$ and we may assume $\alpha(0) = e$.

We can think of $\mathbb{A}^1$ and $\mathbb{A}^1 \setminus \{0\}$ as group varieties.

Then the corollary to Thm 3.4 tells us

- on $(\mathbb{A}^1, +)$ we have $\alpha(x+y) = \alpha(x) + \alpha(y)$
- on $(\mathbb{A}^1 \setminus \{0\}, \cdot)$, we have $\alpha(xy) = \alpha(x) + \alpha(y) + c$ for some $c \in A$.

These can only hold at the same time if α is constant.

$\mathbb{P}^1 \dashrightarrow A$ is constant, since its constant on affine patches.

$\square$

Dfn: • $V/\bar{K}$ is unirational if there is a dominating map $\mathbb{A}^n \dashrightarrow V$, where $n = \dim_{\bar{K}} V$.

• V/K is unirational if $V/\bar{K}$ is.

Proposition 3.10: Every rational map $V \dashrightarrow A$ from V unirational to A an abelian variety is constant.

Proof: WLOG, $K = \bar{K}$.

Think of $\mathbb{A}^n \subseteq \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ open.

Since V is unirational, we get

$$\beta : \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \dashrightarrow V \dashrightarrow A$$

which extends to $\beta : \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \longrightarrow A$.

Then by Corollary 1.5, there exist regular maps $\beta_i : \mathbb{P}^1 \longrightarrow A$ such that $\beta(x_1, \ldots, x_n) = \sum \beta_i(x_i)$ and by Proposition 3.9 each β_i map is constant.

$\square$