Ricky Magner
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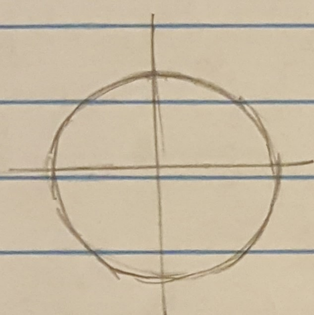
BUNTES

Theorem of the Cube

Crash course in Line Bundles



$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^1$$
$$(x,y) \longmapsto x^2+y^2-1$$

Consider $S = \{ \underline{x} \mid f(\underline{x}) = 0 \} \subseteq \mathbb{R}^2$

Then S is a closed submanifold.

__Q.__ Do all submanifolds arise in this way?

__Example:__ $X = \mathbb{P}^n(\mathbb{C})$

The answer here is __no!__

(Because any $f : X \to \mathbb{C}^1$ is constant).

__Example:__ $X = \mathbb{P}^1_{\mathbb{C}}$, $\mathcal{O}_X$ = structure sheaf on X

$$X = \underset{\substack{\shortparallel \\ (\mathbb{A}^1, t)}}{U_0} \cup \underset{\substack{\shortparallel \\ (\mathbb{A}^1, s)}}{U_1} \qquad \text{on } U_0 \cap U_1, \quad t = s^{-1}.$$

What is a global section of $\mathcal{O}_X$?

$\mathcal{O}_X(U_0) = k[t]$, $\mathcal{O}_X(U_1) = k[s]$.

So, given $f(t) \in k[t]$, $g(s) \in k[s]$,
these glue to a global function iff $f(t) = g(1/t)$.

Since these are polynomials, f and g must be constants.

<u>Dfn</u>: A <u>line bundle</u> $\mathcal{L}$ on X (any scheme) is a locally free $\mathcal{O}_X$-module of rank 1.
ie. There exists an open cover $\{U_i\}$ of X along with an isomorphism $\varphi_i : \mathcal{L}|_{U_i} \xrightarrow{\sim} \mathcal{O}_X|_{U_i}$.

<u>Exercise</u>: (Alternative definition)
A line bundle on X is equivalent to the following data:

- an open cover $\{U_i\}$ of X
- transition maps $\tau_{ij} \in GL_1(\mathcal{O}_X(U_i \cap U_j))$ satisfying $\tau_{ij}\tau_{jk} = \tau_{ik}$ for all $i,j,k \in I$ and $\tau_{ii} = id$.

<u>Example</u>: On $X = \mathbb{P}^n_k$, we have line bundles $\mathcal{O}(d)$ $\forall d \in \mathbb{Z}$.
$\mathcal{O}(d)$ is given by

- $\{U_i\}$ open cover by $U_i \cong \mathbb{A}^n$
- τ_{ij} is given by multiplication by $\left(x_i/x_j\right)^d$.

Exercise: $H^0(X, \mathcal{O}(d))$ (ie. $\Gamma(X, \mathcal{O}(d))$)

$=$ k-vector space spanned by degree d homogenous polynomials in $k[x_0, \dots, x_n]$.

Exercise: All line bundles on $\mathbb{P}^n$ are isomorphic to some $\mathcal{O}(d)$.

We say a line bundle $\mathcal{L}$ on X is trivial if $\mathcal{L} \simeq \mathcal{O}_X$.

Given $\mathcal{L}_1$ and $\mathcal{L}_2$ line bundles on X, we can create a new line bundle

$$\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2.$$

So line bundles on X with $\otimes$ up to isomorphism form a group, denoted $\mathrm{Pic}(X)$:

- Identity: $\mathcal{O}_X$
- Inverses: $\mathcal{L}^{-1} = \mathrm{Hom}(\mathcal{L}, \mathcal{O}_X)$.

Example: On $\mathbb{P}^n_k$, $\mathcal{O}(d_1) \otimes \mathcal{O}(d_2) \simeq \mathcal{O}(d_1 + d_2)$.
Thus

$$\mathrm{Pic}(\mathbb{P}^n_k) \simeq \mathbb{Z}.$$

Fact: If $f: X \longrightarrow Y$, then given $\mathcal{L}$ on Y we can pull back to a line bundle $f^*\mathcal{L}$ on X (dfn omitted).
We also know f^* commutes with $\otimes$, so in fact (as $f^*\mathcal{O}_Y = \mathcal{O}_X$) we get a homomorphism $f^*: \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$.

Relation to (Weil) Divisors

Let X be a normal variety.

This means the stalks of the structure sheaf are integrally closed.
It's just a technical condition that allows us to write down some facts about principal bundles.

Call $Z \subseteq X$ a closed subvariety of codimension 1 a prime divisor.
A divisor on X is a formal sum

$$D = \sum n_Z \cdot Z$$ of prime divisors.

Let $K = K(X)$ be the function field of X (that is, the set of rational maps $f: X \dashrightarrow \mathbb{A}^1$).
Given $f \in K^\times$, we can define

$$\mathrm{div}(f) := \sum_{Z \subseteq X} v_Z(f) \cdot Z$$

Given $D \in \mathrm{Div}(X)$, we can define a line bundle $\mathcal{L}(D)$ on X via

$$\mathcal{L}(D)(U) = \{ f \in K^\times \mid (D + \mathrm{div}(f))|_U \geq 0 \} \cup \{0\}.$$

where

$$D|_U = \sum_{Z \cap U \neq \emptyset} n_Z \cdot (Z \cap U).$$

Proposition: The map

$$Cl(X) := \mathrm{Div}(X) / \mathrm{Princ}(X) \xrightarrow{\mathcal{L}(\cdot)} \mathrm{Pic}(X)$$

is an isomorphism.

Theorem of the Cube

Let U, V, W be complete varieties.
If $\mathcal{L}$ is a line bundle on $U \times V \times W$ such that

$$\mathcal{L}|_{\{u_0\}\times V\times W}, \quad \mathcal{L}|_{U\times\{v_0\}\times W} \quad \text{and} \quad \mathcal{L}_{U\times V\times\{w_0\}}$$

are all trivial, then $\mathcal{L}$ is trivial.

Corollary 5.2: Let A be an abelian variety.
Let $p_i : A\times A\times A \longrightarrow A$ be projection onto the ith coordinate, $p_{ij} = p_i + p_j$ and $p_{123} = p_1 + p_2 + p_3$ (so $p_{12}(x,y,z) = x+y$, etc).

Then, for any $\mathcal{L}$ on A, the line bundle

$$M = p_{123}^*\mathcal{L} \otimes p_{12}^*\mathcal{L}^{-1} \otimes p_{13}^*\mathcal{L}^{-1} \otimes p_{23}^*\mathcal{L}^{-1} \otimes p_1^*\mathcal{L} \otimes p_2^*\mathcal{L} \otimes p_3^*\mathcal{L}$$

is trivial.

Pf: Let $m : A\times A \to A$ be the gp operation, and p, q the projections $A\times A \to A$.
Let $\varphi : A\times A \to A\times A\times A$, $(x,y) \mapsto (x,y,0)$.
Then $p_{123}\circ\varphi = m$, $p_{12}\circ\varphi = m$, $p_{23}\circ\varphi = q$, $p_{13}\circ\varphi = p$,
$p_1\circ\varphi = p$, $p_2\circ\varphi = q$, $p_3\circ\varphi = 0$.

Hence the restriction of $\mathcal{M}$ to $A \times A \times \{0\} (\simeq A \times A)$ is

$$m^* \mathcal{L} \otimes m^* \mathcal{L}^{-1} \otimes q^* \mathcal{L}^{-1} \otimes p^* \mathcal{L}^{-1} \otimes p^* \mathcal{L} \otimes q^* \mathcal{L} \otimes \mathcal{O}_{A \times A}.$$

This is trivial, since pullback commutes with tensor product.
Replacing φ with embeddings into different pairs of coordinates, we see $\mathcal{M}$ restricts to a trivial bundle on $A \times \{0\} \times A$ and $\{0\} \times A \times A$.
Thus, by Thm of the Cube, $\mathcal{M}$ is trivial. ∎

Corollary 5.3: Let $f, g, h : V \to A$, A an abelian variety.
Then for any $\mathcal{L}$ on A, the bundle

$$\mathcal{M} = (f+g+h)^* \mathcal{L} \otimes (f+g)^* \mathcal{L}^{-1} \otimes (f+h)^* \mathcal{L}^{-1} \otimes (g+h)^* \mathcal{L}^{-1} \otimes f^* \mathcal{L} \otimes g^* \mathcal{L} \otimes h^* \mathcal{L}$$

is trivial.

Pf: $\mathcal{M}$ is the pullback of the line bundle of Corollary 5.2 by the map

$$(f, g, h) : V \longrightarrow A \times A \times A.$$ ∎

On a we have $n_A : A \to A$ by $n_A(a) = \underbrace{a + \cdots + a}_{n \text{ times}}$, $n \in \mathbb{Z}$.

Note: $2_A : A \xrightarrow{\Delta} A \times A \xrightarrow{m} A$, and the rest can be defined inductively.
Thus n_A are rational maps.

Corollary 5.4: For $\mathcal{L}$ on A, $n_A^* \mathcal{L} \simeq \mathcal{L}^{(n^2+n)/2} \otimes (-1)_A^* \mathcal{L}^{(n^2-n)/2}$

In particular, if $(-1)_A^* \mathcal{L} \simeq \mathcal{L}$ (symmetric) then $n_A^* \mathcal{L} \simeq \mathcal{L}^{n^2}$.
If $(-1)_A^* \mathcal{L} \simeq \mathcal{L}^{-1}$ (antisymmetric), then $n_A^* \mathcal{L} \simeq \mathcal{L}^n$.

Proof: Use Corollary 5.3 with $f = n_A$, $g = 1_A$, $h = (-1)_A$.
So the bundle

$$n_A^* \mathcal{L} \otimes (n+1)_A^* \mathcal{L}^{-1} \otimes (n-1)_A^* \mathcal{L}^{-1} \otimes n_A^* \mathcal{L} \otimes \mathcal{L} \otimes (-1)_A^* \mathcal{L}$$

is trivial; ie.

$$(n+1)_A^* \mathcal{L} \simeq n_A^* \mathcal{L}^2 \otimes (n-1)_A^* \mathcal{L}^{-1} \otimes \mathcal{L} \otimes (-1)_A^* \mathcal{L}. \qquad (+)$$

We proceed by induction:

- $n=1$ is good in statement
- Plugging in $n=1$ to $(+)$, we get
$$2_A^* \mathcal{L} \simeq \mathcal{L}^2 \otimes \mathcal{L} \otimes (-1)_A^* \mathcal{L} \simeq \mathcal{L}^3 \otimes (-1)_A^* \mathcal{L}.$$

Then induct on n in $(+)$. $\square$

Theorem of the Square (5.5)

Let $\mathcal{L}$ be an invertible sheaf (line bundle) on A. Let $t_a : A \to A$ be translation by $a \in A(k)$.
Then

$$t_{a+b}^* \mathcal{L} \otimes \mathcal{L} \simeq t_a^* \mathcal{L} \otimes t_b^* \mathcal{L}.$$

Proof: Use Corollary 5.3 with $f = id$, $g(x) = a$, $h(x) = b$ to get

$$t_{a+b}^* \mathcal{L} \otimes t_a^* \mathcal{L}^{-1} \otimes t_b^* \mathcal{L}^{-1} \otimes \mathcal{L} \quad \text{is trivial.}$$

This follows since g, h are constant and thus pull back trivially. $\square$

Remark: Apply $\otimes \mathcal{L}^{-2}$ to the above equation to get

$$t_{a+b}^* \mathcal{L} \otimes \mathcal{L}^{-1} \simeq (t_a^* \mathcal{L} \otimes \mathcal{L}^{-1}) \otimes (t_b^* \mathcal{L} \otimes \mathcal{L}^{-1})$$

This gives a group homomorphism

$$\begin{aligned} A(k) &\longrightarrow \mathrm{Pic}(A) \\ a &\longmapsto t_a^* \mathcal{L} \otimes \mathcal{L}^{-1} \quad \text{for any } \mathcal{L} \in \mathrm{Pic}(A). \end{aligned}$$