The adventures of BUNTES

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(J.S. Milne in the style of A.A. Milne)

Chapter 1: in which we are introduced to an important homomorphism, review some concepts, and our story begins

Chapter 2: in which Pooh discovers our main theorem

Chapter 3: in which Owl proves that ampleness of $\mathcal{L}$ (iv) implies finiteness of $K(\mathcal{L})$ (ii)

Chapter 4: in which Rabbit sets out on a long journey to prove finiteness of $H(D)$ implies $|2D|$ is basepoint free and gives a finite map $X \longrightarrow \mathbb{P}^N$

Chapter 5: in which Piglet discovers a corollary

Epilogue: in which we might discuss isogenies

$\boxed{\S 1}$ Abelian variety X

A complete group variety

Goal: $X \longrightarrow \mathbb{P}^N$ for some N.

This motivates the study of line bundles.

Last time: Ricky gave Thm of , and Thm of $\square$

For any line bundle $\mathcal{L}$ on X, there is a group homomorphism

$$\Phi_{\mathcal{L}} : X \longrightarrow \mathrm{Pic}(X)$$
$$x \longmapsto T_x^* \mathcal{L} \otimes \mathcal{L}^{-1}$$

<u>Caution</u>: T_x^* is translation by $-x$, by convention.
(Except in the elliptic curve case)

<u>Example</u>: $X = \mathcal{E}$, an elliptic curve, $\mathcal{L} = L((0))$.

$$\phi_{(0)} : \mathcal{E} \longrightarrow \mathrm{Pic}(\mathcal{E})$$
$$x \longmapsto (x) - (0)$$

In this case, this is in $\mathrm{Pic}^0(\mathcal{E}) \simeq \mathcal{E} \simeq \hat{\mathcal{E}}$ <u>and</u> this is translation invariant.

<u>Pf</u>: Translate by $q \in \mathcal{E}$, $(x+q) - (q)$

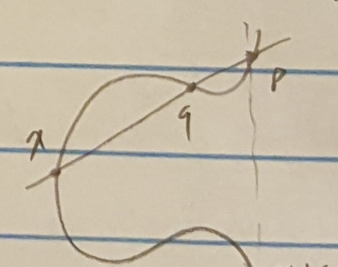
Let p be the third point on the line with x, q

① $(x) + (p) + (q) \simeq 3(0)$

② $(x+q) + (p) \simeq 2(0)$

②-①: $(x+q) - (x) - (q) \simeq -(0)$

$(x+q) - (q) \simeq (x) - (0)$



□

Let $K(\mathcal{L}) = \{x \in X \mid T_x^* \mathcal{L} \simeq \mathcal{L}\}$

Proposition: $K(\mathcal{L})$ is Zariski closed in X.

Proof: Consider $m^* \mathcal{L} \otimes p_2^* \mathcal{L}^{-1}$ on $X \times X$.

Then we consider

$$\{x \in X \mid m^* \mathcal{L} \otimes p_2^* \mathcal{L}^{-1}\}.$$

This is a closed condition.

By the Seesaw principle, restriction is pullback.

Thus the above set is precisely $K(\mathcal{L})$. $\square$

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§2 Let X be an abelian variety.

Let $\mathcal{L} = \mathcal{L}(D)$ be a line bundle.

The following are equivalent:

(i) $H(D) = \{x \in X \mid T_x^* D = D\}$ is finite

(ii) $K(\mathcal{L}) = \{x \in X \mid T_x^* \mathcal{L} \simeq \mathcal{L}\}$ is finite

(iii) $|2D|$ is basepoint free and defines a finite morphism $X \to \mathbb{P}^N$ (not an immersion)

(iv) $\mathcal{L}$ is ample.

Proof strategy: (iii) $\Rightarrow$ (iv) by general algebraic geometry

(ii) $\Rightarrow$ (i) since $K(\mathcal{L})$ is a priori larger than $H(D)$

We do: (iv) $\Rightarrow$ (ii)

(i) $\Rightarrow$ (iii)

$\boxed{\S 3}$ Pf of (iv) $\Rightarrow$ (ii): For contradiction, assume L ample and $K(\mathcal{L})$ is ∞.

Let Y be the connected component of 0 in $K(\mathcal{L})$, $\dim Y > 0$.

Idea: Show trivial bundle is ample on $Y \Rightarrow Y$ is affine.
But, $Y \subseteq X$ closed $\Rightarrow$ complete. Contradiction.

$\mathcal{L}|_Y$ is ample.
$[-1]^* \mathcal{L}|_Y$ is ample.
$\mathcal{L}|_Y \otimes [-1]^* \mathcal{L}|_Y$ is ample.

Consider $d: Y \to Y \times Y$, then $m \circ d =$ constant
$y \mapsto (y, -y)$

$\Rightarrow d^* m^* (\mathcal{L}) = \mathcal{O}_Y$, but $d^* m^* (\mathcal{L}) = \mathcal{L}|_Y \otimes [-1]^* \mathcal{L}|_Y$

$\Rightarrow \mathcal{O}_Y$ is ample. $\square$

$\boxed{\S 4}$ Note: $|2D|$ is always basepoint free.

Thm of $\square$: $T_{x+y}^* D + D \simeq T_x^* D + T_y^* D$

Letting $y = -x$, $2D \simeq T_x^* D + T_{-x}^* D$

Assume D is effective.
For any $y \in X$, choose some x such that RHS doesn't contain y.
Let $E = 2D$.

Let $|E| = |2D|$ (ie. the linear system corresponding to $2D$).
We get

$$\psi_E : X \longrightarrow \mathbb{P}^N \quad \text{(construction to follow).}$$

Can we make this finite?
If ψ_E is not finite, then $\psi(C) = \{pt\}$ for some irreducible curve C.
(Zariski's main theorem).

For each divisor in $|E|$, either it contains C or fails to intersect C.
By changing E if necessary, assume $E \cap C = \phi$.

<u>Claim</u>: $T_x^* E \cap C = 0$ <u>or</u> all of C for all $x \in X$.

<u>Pf 1</u>: Intersection #s are constant. □

<u>Pf 2</u>: $\mathcal{O}(T_x^* E)|_{\tilde{C}}$, when $x = 0$ this is trivial, so $\deg = 0$.

normalisation of C

So $\deg = 0$ for all line bundles.
E effective $\Rightarrow$ $C \cap T_x^* E = \phi$ for all x such that the intersection is not in C. □

<u>Claim</u>: E is invariant by translation by $x - y$ for $x, y \in C$.

<u>Pf</u>: If $e \in E$, $T_{x-e}^*(E) \cap C \neq \phi$
Remember that this is <u>subtraction</u>, so x is in it $(x - (x-e) = e)$.
Because it's not ϕ, it's all of C, so y is in it.

$y - (x-e) \in E$
$= e - (x-y)$

Thus E is invariant under T_{x-y}. □

Now assume (i), that is $H(D) = \{x \in X \mid T_x^* E = E\}$ is finite.

However, if $\varphi_E(C) = \{pt\}$, then $T_{x-y}^*(E) = E$ for all $x, y \in C$.
$\Rightarrow$ $H(D) \supseteq C$ and is not finite.
This is a contradiction.
$\Rightarrow$ φ_E cannot collapse a curve.
$\Rightarrow$ φ_E is finite.

$\boxed{\S 5}$ Corollary: Abelian Varieties are projective.

Proof: Let X be an abelian variety.
Let $U \subseteq X$ be an open affine, $0 \in U$.
Then

$$X \setminus U = D_1 \cup \dots \cup D_t \quad \text{irreducible components.}$$

$$D = \sum D_i.$$

Claim: $H(D) = \{x \in X \mid T_x^* D = D\}$ is finite.

Subpf: If $H(D) \subseteq U$, then $H(D)$ is a complete (closed $\subseteq X$) subset of an affine, thus finite.

If $x \in H$, then $-x \in H$.

Subclaim: If $x \in H$, then T_x^* preserves $\mathcal{U}$.

Subsubpf: If not, let $u \in \mathcal{U}$.

Suppose $u - x = d$ for some $d \in D$.

Then $u = d + x$

However, since $x \in H$, $-x \in H$.

So D invariant under $H \Rightarrow d + x \in D$.

So $u \in D$, contradiction.

So T_x^* preserves $\mathcal{U}$. subsub $\square$

In particular, $0 \in \mathcal{U}$ and $\forall x \in H$, $0 - x \in \mathcal{U}$

$0 + x \in \mathcal{U}$.

Thus $H \subseteq \mathcal{U}$.

Thus H is finite. sub $\square$

Thus, by the main theorem, X is projective. $\square$

Corollary: Abelian varieties are divisible.

$X[n]$ is finite for $n \geq 1$.

Pf: $[n] : X \to X$.

$X[n] = \ker([n])$.

Note that for $x \in X[n]$,

$[n] \circ T_x = [n]$ $\quad (n(y-x) = ny - nx = ny)$.

So for all $\mathcal{L} \in \mathrm{Pic}(X)$,

$$T_x^*([n]^*\mathcal{L}) \simeq [n]^*\mathcal{L}.$$

$\Rightarrow X[n] \subseteq K([n]^*\mathcal{L})$

X projective $\Rightarrow$ There exists $\mathcal{L}$ ample.

Thm of the $\Rightarrow$ $[n]^*\mathcal{L} \simeq \mathcal{L}^{\frac{n^2+n}{2}} \otimes \mathcal{L}^{\frac{n^2-n}{2}}$

Thus $[n]^*\mathcal{L}$ ample.

$\Rightarrow K([n]^*\mathcal{L})$ finite.

$\Rightarrow X[n]$ finite. $\square$

<u>Dfn</u>: Given $f: X \to Y$ a morphism of varieties, get a field extension $K(X)/f^*K(Y)$. If $\dim X = \dim Y$ and f surjective then this is a finite field extension, and $\deg f = d = [K(X) : f^*K(Y)]$.

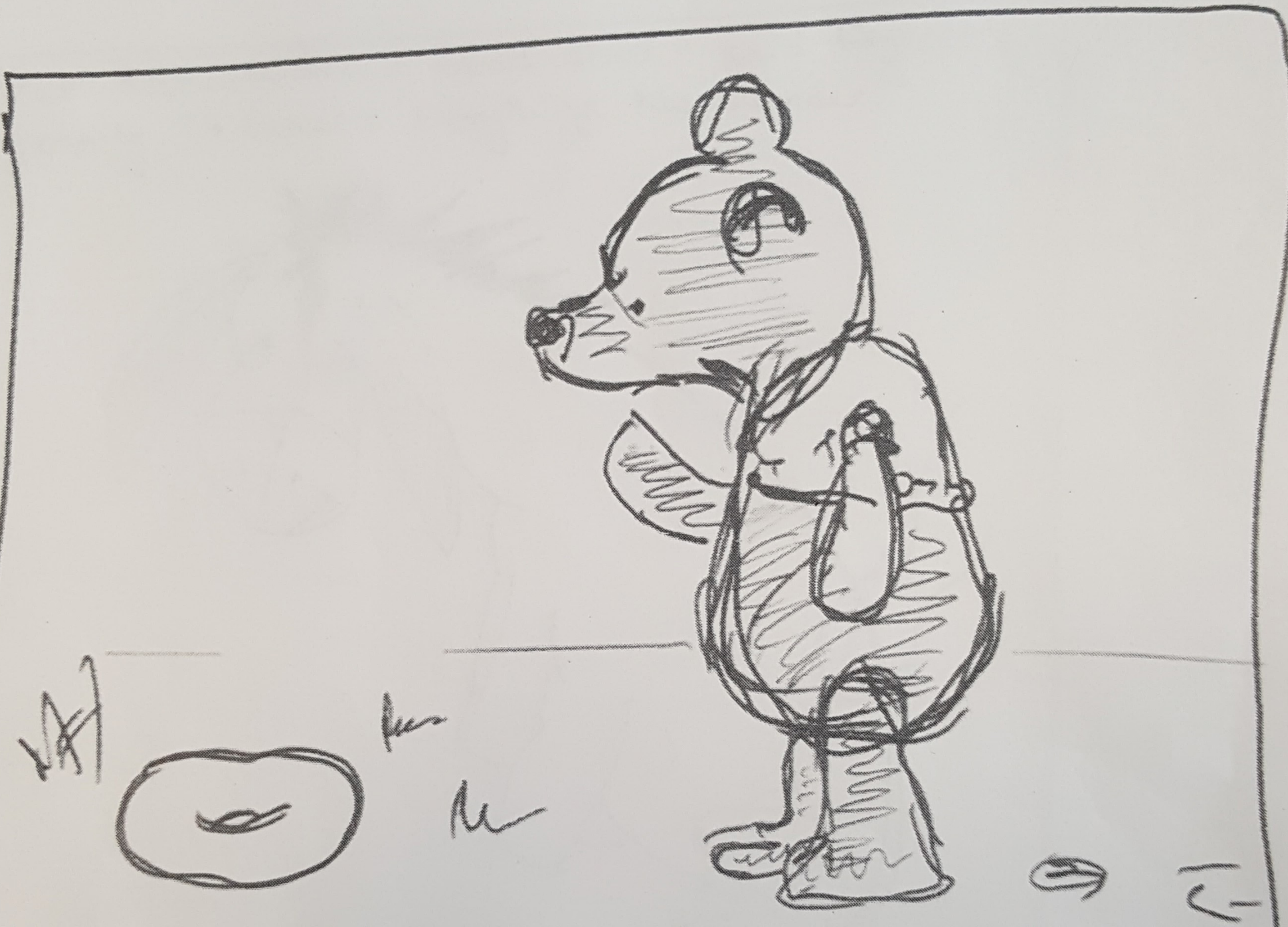
$d = \# f^{-1}(y)$ for almost all y.

<u>Dfn</u>: A homomorphism of abelian varieties $f: X \to Y$ is an <u>isogeny</u> if f is surjective with finite kernel.

Corollary: Deg of $[n]$ is n^{2g} if n is prime to char(k), $k=\bar{k}$, $g=\dim X$.

Pf: D' = ample symmetric divisor, eg. $D' + [-1]^* D' =: D$
We know $[n]^* D \sim n^2 D$

$$\deg\big([n](\underbrace{D \cdots D}_{g})\big) = \big([n]^*D \cdots [n]^*D\big)$$
$$= \big(n^2 D \cdots n^2 D\big)$$
$$= n^{2g}(D \cdots D). \qquad \square$$



"What if the divisor is non effective?" said Winnie the Pooh. "Oh bother."

Jeff Hicks

"That Accounts for a Good Deal," said Eeyore gloomily. "It Explains Everything. No Wonder."

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