Line Bundles and the Dual Abelian Variety

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Meta-goal: Understand line bundles on abelian varieties

Setup: A an abelian variety $/k$.

Last time: For L a line bundle on A, we get a map

$$\phi_L : A(k) \longrightarrow \mathrm{Pic}(A)$$
$$a \longmapsto t_a^* L \otimes L^{-1}$$

where

$$\mathrm{Pic}(A) = \{\text{line bundles on } A\}/\sim.$$

This is a group homomorphism (by the Thm of the $\square$).
We define

$$K(L)(k) = \ker(\phi_L)$$
$$= \{a \in A(k) \mid t_a^* L \simeq L\}.$$

Today: We are going to package these into a big map

$$\phi : \mathrm{Pic}(A) \longrightarrow \mathrm{Hom}(A(k), \mathrm{Pic}(A)).$$
$$L \longmapsto \left(\begin{array}{l} \phi_L : A(k) \longrightarrow \mathrm{Pic}(A) \\ \qquad a \longmapsto t_a^* L \otimes L^{-1} \end{array} \right)$$

Proposition: ① ϕ is a group homomorphism

② $\phi_{t_a^* L} = \phi_L$

Proof: ① $\phi_{L\otimes M}(a) = t_a^*(L\otimes M)\otimes(L\otimes M)^{-1}$
$= (t_a^*L\otimes L^{-1})\otimes(t_a^*M\otimes M^{-1})$
$= \phi_L\otimes\phi_M$

② $\phi_{t_b^*L}(a) = t_a^*(t_b^*L)\otimes(t_b^*L)^{-1}$
$= t_{a+b}^*L\otimes(t_b^*L)^{-1}$
$= t_a^*L\otimes \cancel{t_b^*L}\otimes L^{-1}\otimes \cancel{(t_b^*L)^{-1}}$ (Thm of □)
$= \phi_L(a)$.

□

Dfn: $\mathrm{Pic}^0(A) = \ker(\phi)$
$= \{L\in \mathrm{Pic}(X) \mid \phi_L = 0\}$
$= \{L\in \mathrm{Pic}(X) \mid t_a^*L\cong L \quad \forall a\in A(k)\}$
$= \{\text{translation invariant line bundles}\}/\sim$

Goals: Study $\mathrm{Pic}^0(A)$
Give it an abelian variety structure
Solve a moduli problem
Demonstrate some duality.

§1) See-saws

Thm (See-saw Principle)

Let X, T be varieties, X complete.
Let L be a line bundle on $X \times T$.
Let

$$T_1 = \{ t \in T \mid L|_{X \times \{t\}} \text{ is trivial on } X \times \{t\} \}.$$

Then T_1 is closed in T.
Further, let $p_2 : X \times T_1 \longrightarrow T_1$.
Then $L|_{X \times T_1} \simeq p_2^* M$ for some line bundle M on T_1.

Remark: In fact, $M = p_{2*} L$.

Corollary (That no one states)

Let X, T as above and L, M line bundles on $X \times T$ such that

- $L|_{X \times \{t\}} \simeq M|_{X \times \{t\}}$ for all $t \in T$
- $L|_{\{x\} \times T} \simeq M|_{\{x\} \times T}$.

Then $L \simeq M$.

Proof: We have $(L \otimes M^{-1})|_{X \times \{t\}} \simeq \mathcal{O}_{X \times \{t\}}$ $\forall t$.
So, by the See-saw Principle, we have

$$L \otimes M^{-1} \simeq p_2^* N \quad \text{for some } N.$$

In fact, $N = p_{2*}(L \otimes M^{-1})$.

However

$$(L \otimes M^{-1})_{\{0\} \times T} \simeq \mathcal{O}_T$$

$\Rightarrow \quad N \simeq \mathcal{O}_T$

$\Rightarrow \quad L \otimes M^{-1} \simeq p_2^* \mathcal{O}_T = \mathcal{O}_{X \times T}$

$\Rightarrow \quad L \simeq M.$ $\square$

So to specify a line bundle, it is enough to specify it on $X \times \{t\}$ $\forall t \in T$ and on $\{0\} \times T$.

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§2. Properties of $\mathrm{Pic}^\circ(A)$

Lemma: Let $L \in \mathrm{Pic}^\circ(A)$, $m, p_1, p_2 : A \times A \to A$.

① $m^* L \simeq p_1^* L \otimes p_2^* L$

② Given $f, g : X \to A$, $(f+g)^* L \simeq f^* L \otimes g^* L$

③ $[n]^* L \simeq L^{\otimes n}$

④ If L is nontrivial, $H^i(A, L) = 0$ $\forall i$

⑤ $\phi_L(A(k)) \subseteq \mathrm{Pic}^\circ(A)$.

Proof: ① $\left(m^* L \otimes (p_1^* L)^{-1} \otimes (p_2^* L)^{-1}\right)\big|_{A \times \{a\}} \simeq t_a^* L \otimes L^{-1} \simeq \mathcal{O}_A$

$\left(\text{———} \quad '' \quad \text{———} \right)\big|_{\{0\} \times A} \simeq \mathcal{O}_A$

$\Rightarrow \quad m^* L \otimes (p_1^* L)^{-1} \otimes (p_2^* L)^{-1} \simeq \mathcal{O}_{A \times A}$, by Seesaw.

② $(f+g)^* L = (f \times g)^* m^* L$

$= (f \times g)^* (p_1^* L \otimes p_2^* L) = f^* L \otimes g^* L$

③ By induction using ②

$$[n+1]^*L = [n]^*L \otimes L$$
$$\simeq L^{\otimes n} \otimes L \simeq L^{\otimes n+1}$$

④ If $H^0(A,L) \neq 0$, we have a nontrivial section s of L.
Then $(-1)^*s$ is a nontrivial section of $(-1)^*L \simeq L^{-1}$ (by ③).
If L and L^{-1} have a nontrivial section, then $L \simeq \mathcal{O}_A$.
Since L nontrivial, must have $H^0(A,L) = 0$.

Proceed by induction.
Assume $H^i(A,L) = 0 \ \forall i < j$.
Consider

$$A \xrightarrow{id \times 0} A \times A \xrightarrow{m} A$$
$$a \longmapsto (a,0) \longmapsto a$$

This gives

$$\begin{tikzcd} H^j(A,L) \arrow[r] \arrow[rr, bend right, "id"] & H^j(A \times A, m^*L) \arrow[r] & H^j(A,L). \end{tikzcd}$$

We have

$$H^j(A\times A, m^*L) \simeq H^j(A\times A, p_1^*L \otimes p_2^*L) \quad \text{(by ①)}$$
$$\simeq \underbrace{\bigoplus_{i=0}^{j} H^i(A,L) \otimes H^{j-i}(A,L)}_{=0} \quad \text{(by Künneth formula)}$$

by inductive hypothesis.

$\Rightarrow$ $id: H^j(A,L) \to H^j(A,L)$ factors through 0.
$\Rightarrow$ $H^j(A,L) = 0$.

(5) $\phi_{t_a^*L \otimes L^{-1}} \simeq \phi_{t_a^*L} \otimes \phi_{L^{-1}} = \phi_L \otimes \phi_L^{-1} = 0.$

$\square$

So by (5) we think of ϕ_L as a map

$$\phi_L : A(k) \longrightarrow \mathrm{Pic}^0(A),$$

with kernel $K(L)(k)$.

Theorem: If $K(L)(k)$ is finite, then ϕ_L is surjective.

Proof (Sketch): We have the Mumford Bundle

$$\Lambda(L) = m^*L \otimes p_1^*L^{-1} \otimes p_2^*L^{-1}.$$

(Recall earlier we had $\Lambda(L)$ trivial if $L \in \mathrm{Pic}^0(A)$).

Let $N = \Lambda(L) \otimes p_1^*M^{-1}$ for some $M \in \mathrm{Pic}^0(A)$.

Then

$$N|_{\{a\}\times A} \simeq t_a^*L \otimes L^{-1}, \quad N|_{A\times\{a\}} \simeq t_a^*L \otimes L^{-1} \otimes M^{-1}.$$

Assume $M \notin \mathrm{im}(\phi_L)$.

Then $N|_{A\times\{a\}}$ nontrivial.

$$\Rightarrow H^i(A, N|_{A\times\{a\}}) = 0 \quad \forall i, \ \forall a \in A(k)$$

(Leray Spectral Sequence?) $\Rightarrow H^i(A \times A, N) = 0 \quad \forall i$

If $a \notin K(L)(k)$, $N|_{\{a\}\times A} \simeq t_a^* L \otimes L^{-1}$ nontrivial.

$\Rightarrow H^i(A, N|_{\{a\}\times A}) = 0 \quad \forall i$

So $H^i(A, N|_{\{a\}\times A})$ supported on $K(L)(k)$, a finite set.

(Leray Spectral Sequence?) $\Rightarrow$ In fact $H^i(A, N|_{\{a\}\times A}) = 0 \quad \forall i \ \forall a \in A(k)$.

However, we then have $H^0(A, N|_{\{0\}\times A}) = H^0(A, \mathcal{O}_A) = 0$.

Contradiction!

Thus, cannot chose $M \notin \mathrm{im}(\phi_L)$. $\square$

<u>Remark</u>: On the level of abelian groups, by the first isomorphism theorem, we now have

$$A(k)/K(L)(k) \simeq \mathrm{Pic}^0(A)$$

for any L such that $K(L)(k)$ is finite (eg. L ample).

<u>Idea</u>: Use this to give $\mathrm{Pic}^0(A)$ an abelian variety structure.

First we need to get some more structure on $K(L)$ (this is unnecessary in the char 0 case).

§3 $K(L)$ as a finite group scheme

Proposition: Let X be a complete variety, Y an arbitrary scheme.
Let L be a line bundle on $X \times Y$.
There exists a unique $Y_1 \subseteq Y$ such that
$f: Z \to Y$ factors through Y_1 $\iff$ There exists a line bundle M on Z such that $p_2^* M \simeq (\mathrm{id} \times f)^* L$.

Proof: [Polishchuk, Prop 9.3], [Mumford III, §10 Prop]. □

Remark: Y_1 is precisely the maximal closed subscheme on which L is trivial.

Corollary: Let A be an abelian variety, L a line bundle.
There exists a subscheme $K(L) \subseteq A$ such that its k-points are $K(L)(k)$ as before.

Proof: Apply the above Proposition to the Mumford bundle $\Lambda(L)$ on $A \times A$.
We have a closed subscheme $A_1 \subseteq A$ with the above universal property.
This is the maximal subscheme such that $\Lambda(L)|_{A \times A_1}$ is trivial. For $a \in A(k)$, $\Lambda(L)|_{A \times \{a\}} \simeq t_a^* L \otimes L^{-1}$.
Thus
$$A_1(k) = \{ a \in A(k) \mid t_a^* L \otimes L^{-1} \text{ is trivial} \} = K(L)(k). \ \square$$

Proposition: $K(L)$ is a subgroup scheme of A.

Pf sketch: Given a scheme Z and morphisms $f, g: Z \to K(L)$ (i.e. Z-points of $K(L)$), we use

$$t_{f+g}^* L \otimes L^{-1} \simeq t_f^* L \otimes L^{-1} \otimes t_g^* L \otimes L^{-1} \qquad \text{(Thm of } \square\text{)}$$

where

$$t_f : A \times Z \longrightarrow A \times Z$$
$$(a, z) \longmapsto (a + f(z), z).$$

This shows $K(L)$ closed under addition in A. $\square$

§4) The Dual Abelian Variety

Theorem: Let A be an abelian variety, L an ample line bundle on A.

Then the quotient scheme $A/_{K(L)}$ exists and is an abelian variety of the same dimension as A.

Proof sketch: (Char 0)

Can just think of $K(L)$ as a finite group

On affine pieces $\mathcal{U} = \mathrm{Spec}(R)$,

we set $\mathcal{U}/_{K(L)} = \mathrm{Spec}(R^{K(L)})$.

ring of invariants

Then glue. [Details in Mumford II, §6 Appendix, Theorem]

It is crucial that the orbits be contained in affine opens.

This is automatic since A is projective

(Since orbit is finite, take complement of some hyperplane missing it). $\square$

Dfn Given an abelian variety A, the <u>dual abelian variety</u> is
$$\hat{A} := A/_{K(L)}$$ for L some ample line bundle on A.

<u>Remarks:</u>
- $\hat{A}(k) = \mathrm{Pic}^0(A)$
- We have an isogeny $\phi_L : A \longrightarrow \hat{A}$.

$\hat{A}$ does more than give a variety which characterises line bundles in $\mathrm{Pic}^0(A)$.
In fact, we get the following.

<u>Theorem</u>: Let A be an abelian variety.
There is a uniquely determined bundle $\mathcal{P}$ on $A \times \hat{A}$, called the <u>Poincaré bundle</u>, such that

① $\mathcal{P}|_{A\times\{x\}} \in \mathrm{Pic}^0(A\times\{x\})$ for all $x \in \hat{A}$
② $\mathcal{P}|_{0\times\hat{A}}$ is trivial.
③ If Z is a scheme with a line bundle R on $X \times Z$ satisfying ① and ②, $\exists ! f : Z \to \hat{A}$ s.t. $(id \times f)^*\mathcal{P} = R$.

That is, $(\hat{A}, \mathcal{P})$ represents the functor

$$Z \longmapsto \left\{ L \in \mathrm{Pic}(A\times Z) \;\middle|\; \begin{array}{l} L|_{A\times\{z\}} \in \mathrm{Pic}^0(A\times\{z\}) \quad \forall z \in Z \\ L|_{\{0\}\times Z} \text{ is trivial} \end{array} \right\}$$

and the Poincaré bundle corresponds to $id_{\hat{A}}$.

Proof sketch: We want to construct $\mathcal{P}$ on $A \times \hat{A}$ by considering $\Lambda(L)$ on $A \times A$ and the projection

$$\pi = id \times \phi_L : A \times A \longrightarrow A \times \hat{A}.$$

Then $\mathcal{P}$ should satisfy

$$\pi^* \mathcal{P} = \Lambda(L).$$

To see that this satisfies ① and ②, let $a \in A$ such that $x = \phi_L(a)$.

The

- $\mathcal{P}|_{A \times \{x\}} = (id \times \phi_L)^* \mathcal{P}|_{A \times \{a\}} = \pi^* \mathcal{P}|_{A \times \{a\}}$
$$= \Lambda(L)|_{A \times \{a\}}$$
$$= t_a^* L \otimes L^{-1} \in \mathrm{Pic}^0(A)$$

- $\mathcal{P}|_{\{0\} \times \hat{A}} = \pi^* \mathcal{P}|_{\{0\} \times A}$
$$= \Lambda(L)|_{\{0\} \times A} = L \otimes L^{-1} = \mathcal{O}_A$$

We omit universal property + uniqueness. □

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§5 | Dual morphisms

Let $f : A \longrightarrow B$ be a homomorphism of abelian varieties.
Let $\mathcal{P}_A, \mathcal{P}_B$ be the Poincaré bundles on $A \times \hat{A}$ and $B \times \hat{B}$.
Consider $M = (f \times id_{\hat{B}})^* \mathcal{P}_B$ on $A \times \hat{B}$.
Then
- $M|_{A \times \{x\}} \in \mathrm{Pic}(A \times \{x\})$
- $M|_{\{0\} \times \hat{B}}$ trivial.

Thus we get by the universal property a unique morphism
$\hat{f} : \hat{B} \longrightarrow \hat{A}$, satisfying $(id_A \times \hat{f})^* \mathcal{P}_A \simeq (f \times id_{\hat{B}})^* \mathcal{P}_B$.

Dfn: Given a homomorphism $f: A \to B$ of abelian varieties, $\hat{f}: \hat{B} \to \hat{A}$ as above is called the dual morphism.

Remark: • $\hat{f}: \hat{B}(k) = \mathrm{Pic}^0(B) \longrightarrow \hat{A}(k) = \mathrm{Pic}^0(A)$

$$L \longmapsto f^*L$$

• $\widehat{[n_A]} = [n_{\hat{A}}]$ $\left(\text{since } [n]^*L \simeq L^{\otimes n}\right)$.

Consider now the Poincaré bundle $\mathcal{P}_{\hat{A}}$ on $\hat{A} \times \hat{\hat{A}}$. $\mathcal{P}_A$ on $A \times \hat{A}$ can be thought of as living on $\hat{A} \times A$. Thus, by the universal property of $\mathcal{P}_{\hat{A}}$ we get a morphism

$$\mathrm{can}_A : A \longrightarrow \hat{\hat{A}}.$$

(In fact, this is an isomorphism).

Lemma: $\phi_{f^*L} = \hat{f} \circ \phi_L \circ f$

Proof:
$$\begin{aligned}\hat{f}(\phi_L(f(a))) &= \hat{f}(t^*_{f(a)} L \otimes L^{-1}) \\ &= f^*(t^*_{f(a)} L) \otimes f^*L^{-1} \\ &= t^*_a(f^*L) \otimes f^*L^{-1} \\ &= \phi_{f^*L}(a).\end{aligned}$$

□

Proposition: If $f: A \to B$ is an isogeny, then $\hat{f}: \hat{B} \to \hat{A}$ is an isogeny.
Further, if $N = \ker f$, then $\hat{N} = \ker \hat{f}$, where $\hat{N}$ is the Cartier dual of of N.

Final concepts...

Dfn:
- A morphism $f: A \to \hat{A}$ is symmetric if $f = \hat{f} \circ \mathrm{can}_A$
- A polarisation is a symmetric isogeny $f: A \to \hat{A}$ such that $f = \phi_L$ for some ample line bundle L on A
- A principal polarisation is a polarisation of degree 1 (ie. an isomorphism $A \xrightarrow{\sim} \hat{A}$).

Remark: Elliptic curves always admit principal polarisations $E \xrightarrow{\sim} \hat{E}$.
Thus, often, when one wishes to correctly generalise a result on elliptic curves to abelian varieties, one instead considers principally polarised abelian varieties.
That is, AVs with a principal polarisation.

Phew! Got there.

Aside: Alternate dfn of $\mathrm{Pic}^0(A)$

Dfn: Let L_1, L_2 be line bundles on an abelian variety A.
L_1 and L_2 are <u>algebraically equivalent</u> if there exits a variety Y with a line bundle L and points $y_1, y_2 \in Y$ such that

$$L|_{A\times\{y_1\}} \cong L_1, \quad L|_{A\times\{y_2\}} \cong L_2$$

Remark: This looks a lot like homotopy.

Proposition: $\mathrm{Pic}^0(A) = \{\text{line bundles on } A \text{ that are algebraically equivalent to } \mathcal{O}_A\}/\sim$

Proof: [Polishchuk, Cor 9.2]. $\square$