BUNTES

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Endomorphisms of the Tate Module

Motivation: Let $V_1 \subseteq \mathbb{P}^n$, $V_2 \subseteq \mathbb{P}^m$

$V_i = V(I_i)$

$I_i \neq 0$, (x), $(y-z)$, ...

A <u>rational map</u> is

$$f: V_1 \longrightarrow V_2$$
$$P \longmapsto [f_1(P) : f_2(P) : \cdots : f_m(P)], \quad f_i \in \bar{K}(V_1).$$

Idea: We are applying a polynomial transformation to send points that vanish with respect to some polynomials to another. This should be difficult.

Upshot: There shouldn't be many rational maps.

Isogeny: Rational + Regular + Homomorphism + Surjective + Finite Kernel + ...

Given A, B, $\# \operatorname{Hom}(A, B) = ?$

Or indeed, $\operatorname{rank}_{\mathbb{Z}} \operatorname{Hom}(A, B) = ?$

Notation: A, B, C, A_i, B_i are all abelian varieties

$\sim$ isogeny / isogenous

$\ell \neq \operatorname{char}(k)$.

Theorem (Poincaré's Complete Reducibility Theorem)

$B \subseteq C \implies B \cap C$ is finite, and $B + C = A$

That is, $\begin{array}{ccc} B \times C & \longrightarrow & A \\ (b, c) & \longmapsto & b+c \end{array}$ is an isogeny.

Proof: Choose $\mathcal{L}$ ample on A.

$$\begin{array}{ccc} B & \xrightarrow{\ i\ } & A \\ \phi_{i^*\mathcal{L}} \downarrow & \circlearrowright & \downarrow \phi_{\mathcal{L}} \\ \hat{B} & \xleftarrow{\ \hat{i}\ } & \hat{A} \end{array}$$

Let C = connected component of $\phi_{\mathcal{L}}^{-1}(\ker \hat{i})$ in A.

$\dim C = \dim(\ker \hat{i}) \geq \dim \hat{A} - \dim \hat{B}$

$= \dim A - \dim B$

Claim: $B \cap C$ is finite

$z \in B \cap \phi_{\mathcal{L}}^{-1}(\ker \hat{i}) \iff T_z^* \mathcal{L} \otimes \mathcal{L}^{-1}\big|_B$ is trivial $\iff z \in K(L|_B)$.

$\mathcal{L}|_B$ is ample $\Rightarrow K(L|_B)$ finite $\Rightarrow B \cap C$ is finite.

$\Rightarrow B \times C \longrightarrow A$ has finite kernel.

Further $\dim(B \times C) = \dim B + \dim C \geq \dim A$

$\Rightarrow$ surjective

$\Rightarrow$ isogeny. $\square$

Dfn: A is *simple* if $B \subseteq A \Rightarrow B = 0$ or $B = A$.

Corollary: $A \sim A_1^{n_1} \times \dots \times A_k^{n_k}$, $A_i \not\sim A_j$ for $i \neq j$ and A_i simple.

Pf: Induction on Poincaré Complete Reducibility. □

Corollary: If A, B simple, $\alpha \in \mathrm{Hom}(A,B) \Rightarrow \alpha$ an isogeny or 0.

Pf: $\alpha(A) \subseteq B \Rightarrow \alpha(A) = B$ or 0.

$\ker(\alpha)$'s connected component containing 0 will be an abelian subvariety.
Call this C. Then $C = 0$ or A.

- $C = A \Rightarrow \alpha = 0$
- $C = 0 \Rightarrow \ker(\alpha)$ is finite.

$\Rightarrow \alpha$ an isogeny or 0. □

Corollary: If A, B are simple and $A \not\sim B$, then $\mathrm{Hom}(A,B) = 0$.

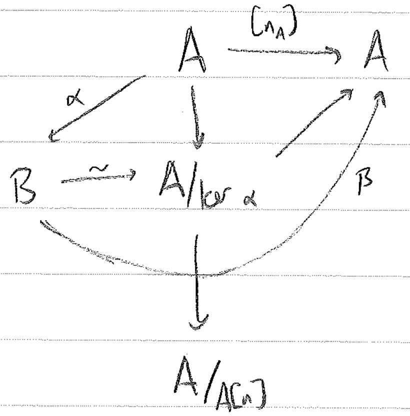
Dfn: $\mathrm{End}^0(A) = \mathrm{End}(A) \otimes \mathbb{Q}$.

<u>Lemma</u>: If $\alpha : A \to B$ is an isogeny, then there exists $\beta : B \to A$ an isogeny such that $\beta \circ \alpha = [n_A]$ for some $n \geq 1$.

<u>Pf</u>: Since α an isogeny, $\ker(\alpha)$ is finite.

$\Rightarrow$ There exists n such that $[n_A] \cdot \ker \alpha = 0$.

$\Rightarrow$ $\ker(\alpha) \subseteq \ker([n_A])$



Then $\beta \circ \alpha = [n_A]$

(Also $\alpha \circ \beta = [n_B]$)

For full generality, replace groups with group schemes. □

<u>Corollary</u>: If A is simple, then $\mathrm{End}^0(A)$ is a division ring.
$(\alpha^{-1} = \beta \otimes \frac{1}{n})$.

<u>Corollary</u> (to Poincaré Complete Reducibility)

If $A \sim A_1^{n_1} \times \cdots \times A_k^{n_k}$ then

$$\mathrm{End}^0(A) = \prod_i \mathrm{End}^0(A_i)^{n_i^2}$$

Proof: $\mathrm{End}(A) \otimes \mathbb{Q} \simeq \left(\prod_{i,j} \mathrm{Hom}(A_i^{n_i}, A_j^{n_j})\right) \otimes \mathbb{Q}$

$$\simeq \left(\prod_i \mathrm{End}(A_i^{n_i})\right) \otimes \mathbb{Q}$$

$$\simeq \prod_i \left(\mathrm{End}(A_i)^{n_i^2} \otimes \mathbb{Q}\right)$$

$$\simeq \prod_i \mathrm{End}^0(A_i)^{n_i^2}. \qquad \square$$

Theorem 7.2: $\dim A = g \Rightarrow \deg [n_A] = n^{2g}$.

Corollary: $\mathrm{char}(k) \nmid n \Rightarrow \ker([n_A]) \simeq (\mathbb{Z}/n\mathbb{Z})^{2g}$.

Pf: If $m | n \Rightarrow |\ker([m_A])| = m^{2g}$.
Then, by the Structure Theorem for f.g. Abelian gps,
$\ker([n_A]) \simeq (\mathbb{Z}/n\mathbb{Z})^{2g}$. $\square$

In particular, $A[\ell^n] = A(k^{sep})[\ell^n]$, then $A[\ell^n] \simeq (\mathbb{Z}/\ell^n\mathbb{Z})^{2g}$.

Dfn (Tate Module): $T_\ell(A) = \varprojlim_n A[\ell^n]$ where $A[\ell^{n+1}] \xrightarrow{[\ell]} A[\ell^n]$.

Proposition: $T_\ell(A) \simeq (\mathbb{Z}_\ell)^{2g}$.

Proof: $(\mathbb{Z}_\ell)^{2g} = \varprojlim_n (\mathbb{Z}/\ell^n\mathbb{Z})^{2g}$. $\square$

$\alpha : A \to B$ induces

$$T_\ell \alpha : T_\ell A \longrightarrow T_\ell B$$
$$(a_1, a_2, \dots) \longmapsto (\alpha(a_1), \alpha(a_2), \dots)$$

<u>Lemma:</u> $\mathrm{Hom}(A, B) \hookrightarrow \underbrace{\mathrm{Hom}(T_\ell A, T_\ell B)}_{\mathbb{Z}_\ell\text{-linear maps}}$

<u>Pf:</u> Let $\alpha \in \mathrm{Hom}(A, B)$ and assume $T_\ell \alpha = 0$.
Let A_i be a simple component of A.
Then $\ker(\alpha|_{A_i}) \supseteq A_i[\ell^n]$ $\forall n$.
$\Rightarrow \alpha = 0$ on A_i for all $A_i \leq A$
$\Rightarrow \alpha = 0$ on A. □

<u>Corollary:</u> $\mathrm{Hom}(A, B)$ is torsion-free.

<u>Recall:</u> $\mathrm{rank}_{\mathbb{Z}} \mathrm{Hom}(A, B) = ?$

So really we want

<u>Thm*:</u> $\mathrm{Hom}(A, B) \otimes \mathbb{Z}_\ell \hookrightarrow \mathrm{Hom}(T_\ell A, T_\ell B)$

Pf*: (Part I)

$$\begin{array}{ccc}
\mathrm{Hom}(A,B)\otimes\mathbb{Z}_\ell & \overset{?}{\hookrightarrow} & \mathrm{Hom}(T_\ell A, T_\ell B) \\
\downarrow\wr & & \downarrow\wr \\
\prod_{i,j}\left(\mathrm{Hom}(A_i,B_j)\otimes\mathbb{Z}_\ell\right) & \overset{?}{\hookrightarrow} & \prod_{i,j}\mathrm{Hom}(T_\ell A_i, T_\ell B_i)
\end{array}$$

So we reduced to showing

$$\mathrm{Hom}(A_i, B_j)\otimes\mathbb{Z}_\ell \hookrightarrow \mathrm{Hom}(T_\ell A, T_\ell B)$$

If $A_i \nsim B_j$, LHS $=0$, so done.

If $A_i \sim B_j$, $\mathrm{Hom}(A_i, B_j) \hookrightarrow \mathrm{End}(A_i)$

So we are reduced to showing
$$\mathrm{End}(A)\otimes\mathbb{Z}_\ell \hookrightarrow \mathrm{End}(T_\ell A) \quad \text{for } A \text{ simple.}$$

[To be continued]

Dfn: Let V/k be a vector space.
A map $f: V \to k$ is called a (homogeneous) polynomial function of degree d if for all $\{v_1, \dots, v_n\} \subseteq V$ linearly independent, $f(\lambda_1 v_1 + \dots + \lambda_n v_n)$ is given by a (homogeneous) polynomial of degree d in λ_i.

Dfn: $\deg : \mathrm{End}(A) \longrightarrow \mathbb{Z}$

$$\alpha \longmapsto \begin{cases} \deg \alpha, & \alpha \text{ an isogeny} \\ 0, & \text{else.} \end{cases}$$

Thm: deg extends uniquely to a polynomial function of degree $2g$ on $\mathrm{End}^0(A) \longrightarrow \mathbb{Q}$.

Pf: Remarkably subtle (see Mike). $\square$

Pf*: [continued]

Wanted to show $\mathrm{End}(A) \otimes \mathbb{Z}_\ell \hookrightarrow \mathrm{End}(T_\ell A)$ for A simple.

This holds $\iff$ for any finitely generated $M \subseteq \mathrm{End}(A)$
$M \otimes \mathbb{Z}_\ell \hookrightarrow \mathrm{End}(T_\ell A)$

Let $M^{div} = \{ f \in \mathrm{End}(A) \mid nf \in M \text{ for some } n \geq 1 \}$

Claim: M^{div} is finitely generated.

Pf of claim: $M^{div} = (M \otimes \mathbb{Q}) \cap \mathrm{End}(A)$ (as subsets of $\mathrm{End}^0(A)$)

$\deg : M \otimes \mathbb{Q} \longrightarrow \mathbb{Q}$ is a polynomial, thus continuous.

$U = \{ \phi \in M \otimes \mathbb{Q} \mid \deg(\phi) < 1 \}$ is open in $M \otimes \mathbb{Q}$.

$U \cap M^{div} = 0$

$\Rightarrow M^{div}$ is a discrete subgroup of the finite dimensional $\mathbb{Q}$-vector space $M \otimes \mathbb{Q}$

$\Rightarrow M^{div}$ is finitely generated. //

Since $M \hookrightarrow M^{div}$, $M \otimes \mathbb{Z}_\ell \hookrightarrow M^{div} \otimes \mathbb{Z}_\ell$

So we are reduced to showing

$M^{div} \otimes \mathbb{Z}_\ell \hookrightarrow End(T_\ell A)$. (i.e. assume $M = M^{div}$)

Let $f_1, \dots, f_r$ be a $\mathbb{Z}$-basis for M and suppose that

$\sum a_i T_\ell(f_i) = 0$ for some $a_i \in \mathbb{Z}_\ell$, not all 0.

We can assume not all a_i are divisible by ℓ.
Choose $a_i' \in \mathbb{Z}$ such that $a_i' \equiv a_i \pmod{\ell}$.

$f = \sum a_i' f_i \in End(A)$

$\sum a_i' T_\ell(f_i) =$ "0 on the first coordinate"
(thinking of $T_\ell A$ as an inverse limit)

$A[\ell] \subseteq \ker f$

$\Rightarrow$ There exists g such that $f = \ell g$.

$f \in M \Rightarrow g \in M^{div} = M$

$\Rightarrow g = \sum b_i f_i \Rightarrow f = \sum \ell b_i f_i = \sum a_i f_i$

So $\ell | a_i \ \forall i$.

This is a contradiction.

Thus the basis $\{f_i\}$ cannot become linearly dependent under T_ℓ.

$\Rightarrow \mathrm{End}(A) \otimes \mathbb{Z}_\ell \hookrightarrow \mathrm{End}(T_\ell(A))$.

$\Rightarrow \mathrm{Hom}(A, B) \otimes \mathbb{Z}_\ell \hookrightarrow \mathrm{Hom}(T_\ell A, T_\ell B)$. $\square$

Thus $\mathrm{rank}_{\mathbb{Z}} \mathrm{Hom}(A, B) \leq \mathrm{rank}_{\mathbb{Z}_\ell} \mathrm{Hom}(T_\ell A, T_\ell B)$

$\leq \mathrm{rank}\, \mathbb{Z}_\ell \mathrm{Hom}(\mathbb{Z}_\ell^{2g_A}, \mathbb{Z}_\ell^{2g_B})$

$\leq 4 \dim A \dim B$.