Polarizations and Étale Cohomology

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Let A/k be an abelian variety (possibly X sometimes)

Defn: A polarisation is an isogeny $\lambda : A \longrightarrow \hat{A}$ such that over $\bar{k}$ we have a line bundle $\mathcal{L}$ such that
$$\lambda = \lambda_{\mathcal{L}} : a \longmapsto t_a^* \mathcal{L} \otimes \mathcal{L}^{-1}.$$

The degree of a polarisation is its degree as an isogeny.

A degree 1 polarisation is a principal polarisation.

Natural question: What can we read off from $\mathcal{L}$ about $\lambda_{\mathcal{L}}$?

To answer this, recall sheaf cohomology.

If we have a short exact sequence of sheaves, then Γ is not necessarily right exact.

Example (classical): $X = \mathbb{C}^\times$
$$0 \longrightarrow \mathbb{Z} \overset{i}{\hookrightarrow} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \longrightarrow 0$$
but
$$0 \longrightarrow \Gamma(X, \mathbb{Z}) \hookrightarrow \Gamma(X, \mathcal{O}_X) \underbrace{\longrightarrow \Gamma(X, \mathcal{O}_X^\times)}_{\text{not surjective, no global } \log(z)}$$

To remedy/control this, we introduce additional "things".

Explicitly, for a sheaf $\mathcal{F}$, take an injective resolution

$$0 \longrightarrow \mathcal{F} \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots \qquad \text{chain complex}$$

apply Γ and truncate

$$0 \xrightarrow{d^0} \Gamma(X, I^0) \xrightarrow{d^1} \Gamma(X, I^1) \xrightarrow{d^2} \cdots$$

and take cohomology

$$H^0(X, \mathcal{F}) = \frac{\ker d^1}{\operatorname{im} d^0}, \ \ldots, \ H^i(X, \mathcal{F}) = \frac{\ker d^{i+1}}{\operatorname{im} d^i}$$

Dfn: The Euler (-Poincaré) characteristic of a line bundle $\mathcal{L}$ is

$$\chi(\mathcal{L}) = \sum (-1)^i \dim_k H^i(X, \mathcal{L}).$$

Thm (Riemann-Roch)

① $\deg(\lambda_{\mathcal{L}}) = \chi(\mathcal{L})^2$

② If $\mathcal{L} = \mathcal{L}(D)$, then $\chi(\mathcal{L}) = \dfrac{(D^g)}{g!}$, where $g = \dim(A)$.

g-fold self intersection number

Thm (Vanishing)

If $\# K(\mathcal{L}) < \infty$ then there is a unique $0 \leq i(\mathcal{L}) \leq g$ with

$$H^{i}(X, \mathcal{L}) \neq 0$$
$$H^{r}(X, \mathcal{L}) = 0 \quad \forall r \neq i.$$

Moreover, $i(L^{-1}) = g - i(\mathcal{L})$.

Recall that if $\mathcal{L}$ is ample, then $\# K(\mathcal{L}) < \infty$.

Moreover, if $\mathcal{L}$ is very ample then $H^0(X, \mathcal{L}) \neq 0$, i.e. $i(\mathcal{L}) = 0$.

Thm (Finiteness)

The set of all Abelian Varieties over a finite field, of dim g, with a polarisation of degree d^2 is finite.

Proof (Super sketch): Over a finite field k, there exists an ample $\mathcal{L}$ with $\lambda_{\mathcal{L}}$ a polarisation of deg d^2.

Using the above, $\chi(\mathcal{L}^3) = 3^g d$, but $\mathcal{L}^3$ is very ample.

So

$$\dim H^0(A, \mathcal{L}^3) = 3^g d.$$

So we get an embedding into $\mathbb{P}^{3^g d - 1}$.

The degree of $A \subseteq \mathbb{P}^{3^g d-1}$

$$((3D)^g) = 3^g d\, g!$$

Such an embedding is determined by its Chow form (some big polynomial) of fixed degree.

Hence the set of such is finite. $\square$

Étale cohomology

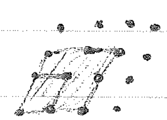
Recall for abelian varieties over $\mathbb{C}$, we had singular cohomology $H^i(A(\mathbb{C}), \mathbb{Z})$.

Want to emulate this over a general field.

We had $\pi_1(A(\mathbb{C}), 0) = \pi^{-1}(0) = \Lambda$

$$\begin{array}{c} V \\ \downarrow \pi \\ A(\mathbb{C}) = V/\Lambda \end{array}$$

Isogenies gave us covers, eg. 4-fold cover by $[2]$.



Problem: Zariski is too coarse.

eg. $[2] : A \to A$, there exists no Zariski open $U \subseteq A$ such that

$$[2]|_U \xrightarrow{\sim} \mathrm{im}([2]|_U).$$

Isogenies are not local isomorphisms for the Zariski topology.

What is a local isomorphism?

Given $f: X \to Y$, we want $U \subseteq Y$ open s.t.

$$\begin{tikzcd} f^{-1}(U) \arrow[r, "p"] \arrow[d] & U \arrow[d, hook, "\text{open inclusion}"] \\ X \arrow[r] & Y \end{tikzcd}$$

$f^{-1}(U)$ disjoint union of schemes which are $\cong U$ by p

If I have a map $f: X \to Y$ which I want to call a local isomorphism, I can adjust my view of open sets.

$$\begin{tikzcd} X \arrow[r, "id"] \arrow[d, "id"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Allow $f: X \to Y$ to be an "open set".

<u>Dfn</u>: Let X, Y be nonsingular varieties $/k = \bar{k}$.
We say $f: X \to Y$ is <u>étale at $P \in X$</u> if
$$df: T_{\text{gt}_P} X \longrightarrow T_{\text{gt}_{f(P)}} Y$$

is an isomorphism.

We say f is <u>étale</u> if it is étale at P for all $P \in X$.

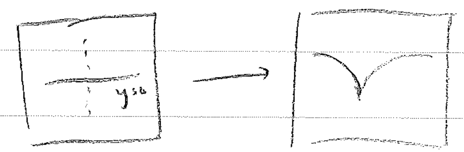
Proposition: If $f : \mathbb{A}^m \longrightarrow \mathbb{A}^m$, with coordinates (x_i) and (y_j), then f is étale at $P = (a_1, \ldots, a_m) \in \mathbb{A}^m$

if the matrix

$$\left(\left(\frac{\partial y_j \circ f}{\partial x_i} \right)_{i,j=1,\ldots,m} \Bigg|_{(a_1,\ldots,a_m)} \right) \quad \text{is nonsingular.}$$

Example: $f : \mathbb{A}^2 \longrightarrow \mathbb{A}^2$

$(x,y) \longmapsto (x^3,\ x^2+y)$



$$\begin{pmatrix} 3x^2 & 0 \\ 2x & 1 \end{pmatrix} \quad \text{is singular when } x=0 \text{ (or char} = 3)$$

Proposition: The maps $[n] : A \longrightarrow A$ are étale if char $k \nmid n$.

Proof: $(d(\alpha+\beta))|_0 = (d\alpha)|_0 + (d\beta)|_0$

So $d[n] = d(\mathrm{id} + \cdots + \mathrm{id})$

$= 1 + \cdots + 1$

$= n$

⇝ k-vector space map

n is an isomorphism precisely when char $k \nmid n$. $\square$

Dfn (Étale morphism)

A morphism $f: X \longrightarrow Y$ of schemes is étale if it is flat and unramified.

All isogenies are finite and flat.

Dfn: Let $\text{FÉt}/X$ be the category of finite étale maps $\pi: Y \longrightarrow X$ (finite étale covers).

If we pick a base point $x \in X$, we can define

$$F: \text{FÉt}/X \longrightarrow \text{Set}$$
$$\pi \longmapsto \mathrm{Hom}_X(x, Y) \quad \left(\text{eg.} \ \approx \pi^{-1}(x) \quad \begin{array}{ccc} & & Y \\ & \nearrow & \downarrow \pi \\ x & \hookrightarrow & X \end{array} \right)$$

This functor is in fact pro-representable, ie. we can find a projective system

$$\tilde{X} = (X_i)_{i \in I} \qquad I \text{ some directed set}$$

such that

$$F(Y) = \mathrm{Hom}_X(\tilde{X}, Y) = \varinjlim_i \mathrm{Hom}_X(X_i, Y).$$

Then

$$\pi_1^{\text{ét}}(X, x) := \mathrm{Aut}_X(\tilde{X}) = \varprojlim_i \mathrm{Aut}_X(X_i).$$

So we need to understand finite étale covers of abelian varieties.

Proposition (Surprising Proposition!): Let X be a complete variety $/k$, $e \in X(k)$ with some map $m : X \times_k X \longrightarrow X$ such that
$$m(e, x) = x = m(x, e).$$
Then (X, m, e) is an abelian variety.

Thm (Lang-Serre): Let X/k be an abelian variety and Y/k such that
$$f : Y \longrightarrow X$$
is a finite étale cover and there exists $e_Y \in Y(k)$ such that $f(e_Y) = e_X$.

Then Y can be given an abelian variety structure such that f becomes a separable isogeny.

Last time we saw that for any isogeny $\alpha : B \longrightarrow A$, there exists $\beta : A \longrightarrow B$ such that $\beta \circ \alpha = [n]$ on A.

$$\begin{tikzcd} A \arrow[dr, "\beta"] \arrow[dd, "{[n]}"'] & \\ & B \arrow[dl, "\alpha"] \\ A & \end{tikzcd}$$

any étale cover

So for our $\tilde{A}$, we can take
$$(A)_{i \in S}$$
with multiplication by n as covering maps.

Thus, $\pi_1(A, 0) = \varprojlim_i \mathrm{Aut}(A)$
$= \varprojlim_n A[n]$

Thm:
- $H^1_{\text{ét}}(A, \mathbb{Z}_\ell) = \mathrm{Hom}(\pi_1(A, 0), \mathbb{Z}_\ell) \cong \mathrm{Hom}(T_\ell A, \mathbb{Z}_\ell)$
- $H^r_{\text{ét}}(A, \mathbb{Z}_\ell) = \Lambda^r H^1_{\text{ét}}(A, \mathbb{Z}_\ell)$.