BUNTES - Dessins d'Enfants

Jan 26, 2018

Introduction

Welcome to the Spring 2018 semester for BUNTES!
Our topic this semester is Dessins d'Enfants.
Thank you all for attending and being involved!
Also, thank you to Alex Best for suggesting the topic.

The plan for today is to give a broad overview of some definitions and topics to frame the rest of the semester.
I will not be giving many details or proofs, but hopefully we will be motivated to come back and explore things in detail.

Bookkeeping:
- This course should be allowed to be taken for credit
- Ali Altuğ is the mentor for the course, though he will not be able to attend the talks
- Please sign up to give talks once the schedule is set up
- Suggest topics/papers that look interesting to you!
- It would be good if each week the speaker came up with some exercises
- Conferences (AWS etc)

§1) Belyi's Theorem

Question: When is an algebraic curve $/\mathbb{C}$ in fact defined over $\overline{\mathbb{Q}}$?

Note that this actually means the curve is defined over K, for some number field $K/\mathbb{Q}$.

Given that we are working in the category of curves $/\mathbb{C}$, it is natural to think of them as Riemann surfaces.

This is a question of great depth, and was of interest to Deligne and Grothendieck.

The answer is surprising.

Theorem [Belyi]: Let $X/\mathbb{C}$ be an algebraic curve. Then

X is defined over $\overline{\mathbb{Q}}$ $\iff$ There exists a holomorphic function $f : X \longrightarrow \mathbb{P}^1\mathbb{C}$ ramified only over $\{0, 1, \infty\}$.

Just a reminder.

Dfn: (Algebraic geometry) A morphism $f : X \to Y$ is *ramified* at $x \in X$ if locally the induced ring morphism $f^{\#} : \mathcal{O}_{Y, f(x)} \longrightarrow \mathcal{O}_{X,x}$ gives an extension of fields

$$\mathcal{O}_{Y, f(x)}/_{m} \longrightarrow \mathcal{O}_{X,x}/_{(f^{\#}(m))}$$

which is *inseperable*.

Dfn: (Riemann Surfaces) A morphism $f: X \to Y$ is ramified at a point $x \in X$ if there exists a choice of charts around x and $f(x)$ such that locally we have $f(x) = x^n$.

A morphism $f: X \to Y$ is ramified over $y \in Y$ if there exists $x \in X$ such that $f(x) = y$ and f is ramified at x.

Dfn: A morphism $f: X \to \mathbb{P}^1\mathbb{C}$ ramified only over $\{0, 1, \infty\}$ is called a Belyi morphism.
A Belyi morphism is called clean (or pure) if the ramification indices over 1 are all exactly 2.

Remark: A clean Belyi morphism is necessarily of even degree.

So our question is reduced to: when does a curve $X/\mathbb{C}$ admit a Belyi morphism?
We can in fact restrict ourselves to studying only clean Belyi morphisms by the following.

Prop: A curve $X/\mathbb{C}$ admits a Belyi morphism
$\iff$ $X/\mathbb{C}$ admits a clean Belyi morphism.

Pf: $(\Rightarrow)$ If $\alpha: X \to \mathbb{P}^1\mathbb{C}$ is a Belyi morphism, then $\beta = 4\alpha(1-\alpha)$ is clean.

$\square$

§2) Dessins d'Enfants

One way to study Belyi morphisms is via Dessins.
Then we defined as follows.

Dfn: A Dessin d'Enfant (or Grothendieck Dessin, or Dessin) is a cellular decomposition (X_0, X_1, X_2) where
X_2 - is a compact Riemann surface
X_1 - is a graph
X_0 - is a finite set of points
such that
$X_2 \setminus X_1$ is a disjoint union of open cells
$X_1 \setminus X_0$ is a disjoint union of line segments.

A Dessin is clean if the vertices, i.e. X_0, can be coloured black + white such that no adjacent vertices have the same colour.

Note: In some references, the language is as follows

Here	Some places
Dessin	Pre-clean dessin
Clean dessin	Dessin

Lemma: The data of a dessin is equivalent to a graph with an ordering on the edges coming out from each vertex.

Dfn: A morphism of dessins is a map $f_2 : X_2 \to X_2'$ such that $f_1(X_1) \subseteq X_1'$ and $f(X_0) \subseteq X_0'$.

Isomorphism classes of dessins are sometimes referred to as abstract dessins.

§3) The Grothendieck Correspondence

How do dessins relate to Belyi morphisms?

Given a (clean) Belyi morphism $f : X \to \mathbb{P}^1\mathbb{C}$, then $f^{-1}([0,1])$ is a (clean) dessin, using the alternative definition as in the lemma.

Theorem [Grothendieck]: The function

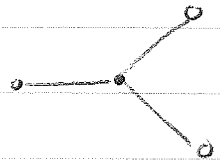
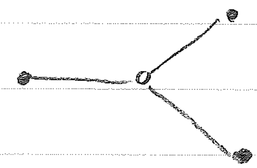
$$\begin{array}{ccc} \{\text{(Clean) Belyi morphisms}\} & \longrightarrow & \{\text{(Clean) dessins}\} \\ (f : X \to \mathbb{P}^1\mathbb{C}) & \longmapsto & f^{-1}([0,1]) \end{array}$$

is a bijection.

When colouring the dessin arising from a Belyi morphism, the standard approach is to colour preimages of 0 black and preimages of 1 white.

Examples: $\mathbb{P}^1\mathbb{C} \longrightarrow \mathbb{P}^1\mathbb{C}$ $\qquad$ $\mathbb{P}^1\mathbb{C} \longrightarrow \mathbb{P}^1\mathbb{C}$

$x \longmapsto x^3$ $\qquad$ $x \longmapsto x^3+1$

§4) Covering spaces and Galois groups

A Belyi morphism determines a cover of $\mathbb{P}^1\mathbb{C} \setminus \{0,1,\infty\}$.
These coverings are in bijection with subgroups of $\hat{\pi}_1(\mathbb{P}^1\mathbb{C}\setminus\{0,1,\infty\})$.

Corollary [Belyi]: $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts faithfully on $\hat{\pi}_1(\mathbb{P}^1\mathbb{C}\setminus\{0,1,\infty\})$.

Pf: By Belyi's Thm, each elliptic curve $E/\overline{\mathbb{Q}}$ admits a Belyi morphism.
Further, for each $j \in \overline{\mathbb{Q}}$ there is an elliptic curve E_j with j-invariant j.
Assume $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts trivially.
Then

$$\sigma(E_j) = E_{\sigma(j)} = E_j \qquad \forall j \in \overline{\mathbb{Q}}$$
$$\Rightarrow \sigma(j) = j \qquad \forall j \in \overline{\mathbb{Q}}$$
$$\Rightarrow \sigma = id. \qquad \square$$

This group is very concrete, $\hat{\pi}_1(\mathbb{P}^1\mathbb{K}\setminus\{0,1,\infty\}) \simeq \widehat{\mathbb{Z}*\mathbb{Z}}$.

Corollary [Grothendieck]: The action of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ on the space of dessins is faithful.

We can often say much more.

For a dessin (X_0, X_1, X_2), let the genus be the genus of the Riemann surface X_2.

Thm: The action of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ on dessins of any fixed genus is faithful.

Our proof above in fact demonstrates the genus 1 argument (combined with the Grothendieck correspondence).

§5) Further topics

- Invariants of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$

  What properties of Dessins are preserved by the Galois action (genus, as above)

- Belyi degrees of curves

  What is the minimal degree of a Belyi morphism a curve admits?

- Computational Aspects

  How does one efficiently implement the Grothendieck correspondence?

- Grothendieck - Teichmüller Theory

  Studying the action of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ on the tower $\hat{\pi}_1(M_{g,n})$ to gain a more concrete understanding of its structure

- More!

<u>Exercises</u>

① Compute the dessins for the following Belyi morphisms:

(a) $\mathbb{P}^1\mathbb{C} \longrightarrow \mathbb{P}^1\mathbb{C}, \quad x \longmapsto x^4$
(b) $\mathbb{P}^1\mathbb{C} \longrightarrow \mathbb{P}^1\mathbb{C}, \quad x \longmapsto x^2(3-2x)$
(c) $\mathbb{P}^1\mathbb{C} \longrightarrow \mathbb{P}^1\mathbb{C}, \quad x \longmapsto \dfrac{1}{x(2-x)}$

② Give an alternate proof of the fact that
(X admits a Belyi morphism $\Longleftrightarrow$ It admits a clean one)
using dessins and the Grothendieck correspondence.

③ Prove that the Belyi morphism corresponding to a tree that sends ∞ to ∞ is a polynomial.