BUNTES

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Dessins, Points on Mordell Curves & a Proof of ABC

ABC for Polynomials

Let $e, f, g \in K[x]$, $K = \bar{K}$, $\mathrm{char}(K) = 0$.

Let

$$\mathrm{rad}(e) = \prod_{\substack{p|e \text{ prime} \\ (\text{i.e. irred})}} p, \qquad r(e) = \deg(\mathrm{rad}(e)) = \#\{x \in K \mid e(x) = 0\}$$

$$h(e) = \deg(e)$$

$$h(\{e, f, g\}) = \max\{h(e), h(f), h(g)\}$$

Thm (ABC): ① If e, f, g are pairwise coprime and $e + f = g$.

Then

$$h(e, f, g) < r(e, f, g) = r(e) + r(f) + r(g)$$

② Noting that $h(\ldots), r(\ldots) \in \mathbb{Z}$, we see that the above inequality is sharp if $h(e, f, g) = r(e, f, g) - 1$.

The inequality is sharp $\Leftrightarrow$ f/g is Belyi.

Proven originally by Mason-Stothers.

We in fact normalise so that $\left(\frac{f}{g}\right)(\infty) \in \{0, 1, \infty\}$.

The content of this is to say that if f and g have the same degree, then they have the same leading coefficients

(In particular, the degree of e will be lower).

Pf: ① Let $\phi = f/g$. $\qquad \deg(\phi) = \max\{\deg f, \deg g\}$

$$= h(e,f,g) =: h$$

View this as $\phi: \mathbb{P}^1 \longrightarrow \mathbb{P}^1$.

Apply Riemann-Hurwitz.

$$-2 = -2h + \sum_p e_\phi(p) - 1$$

Let $R_x = \sum_{p:\phi(p)=x} e_\phi(p) - 1$

R_1: $\left(\frac{f}{g}\right)(x) = 1 \Rightarrow e(x) = 0 \Rightarrow R_1 = h(e) - r(e)$

R_0: $\left(\frac{f}{g}\right)(x) = 0 \Rightarrow f(x) = 0$ or $g(x) = \infty$

$g(x) = \infty \Rightarrow x = \infty$ (g is a polynomial)

However, we can rearrange the equation $e+f=g$ to ensure

$h(f) \geq h(g)$

In which case $g(x) = \infty \Rightarrow \left(\frac{f}{g}\right)(x) \Rightarrow \infty$

So we can ignore this case.

Thus $\frac{f}{g} = 0 \Rightarrow f(x) = 0 \Rightarrow R_0 = h - r(f)$

R_∞: $\left(\frac{f}{g}\right)(x) = \infty \Rightarrow g(x) = 0$ or $x = \infty$

If $h(f) = h(g) \Rightarrow \phi(\infty) \neq \infty$

$\Rightarrow R_\infty = 1 - r(g)$

In Riemann-Hurwitz,

$$-2 = -2h + h + h + h(e) - r(e) - r(f) - r(g) + \underbrace{R}_{\text{leftover ramification}} - \delta_{h(f)>h(g)}$$

$$R = h - r(efg) - 2 + \delta_{h(f)>h(g)}$$

$$R \ge 0 \quad \Rightarrow \quad h \ge \begin{cases} r(ef_g)+1, & \text{if } h(f) > h(g) \\ r(ef_g)+2, & \text{if } h(f) = h(g) \end{cases}$$

② Equality exactly when $R=0 \Leftrightarrow \frac{f}{g}$ is Belyi. □

One can read the proof also in Washi Golding, Lang Memorial Volume.

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§2. Let's find integer points on

$$E_k: \ y^2 = x^3 + k$$

Example [Euler]: $y^2 = x^3+1$ only when $(2,3)$

Impressive party trick: $y^2 = x^3 - (5009^3 - 1001^2)$
has integer point $(5009, 1001)$

Not actually impressive, as $|x|^3 < |k|$.

How far can we go?
Baker gives the bound for $(x,y) \in E_k(\mathbb{Z})$, $\max(x,y) \le e^{10^{10}|k|^{1000}}$.

Switch to minimizing $x^3 - y^2$ for integers x, y.

Conjecture [Marshall Hall]: If $x^3 - y^2 = k$, then $|k| > \sqrt{|x|}/5$

The 5 in the denominator came from info at the time and this conjecture is in fact false.

Elkies has a counterexample (with enormous coefficients).
To account for his example, you'd need to replace 5 by 46.6.

How do we find triples (x, y, k) as above?

One approach: Find $X(t)$, $Y(t)$, $K(t)$ with $X(t)^3 - Y(t)^2 = K(t)$.
Want to minimize $\deg K(t)$.
Apply Mason-Stothers (ABC for polynomials).

Let $6m = \deg X(t)^3$, so $\deg X = 2m$, $\deg Y = 3m$

$$6m < 2m + 3m + \deg k$$
$$\Rightarrow \deg k \geq m+1.$$

We have just proved part ① of

Conjecture [Birch-Chowla, Hall, Schinzel, 65] ① Let X, Y be coprime polynomials such that $\deg X = 2m$, $\deg Y = 3m$, ~~and~~ X^3, Y^2 have equal leading coefficient.

Then $\deg(X^3 - Y^2) \geq m+1$

② This bound is sharp.

Part ② was proven by Stothers in '81.

Sharpness $\Leftrightarrow$ $\frac{X(t)^3}{K(t)} : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ is Belyi

$$= \frac{K(t) + Y(t)^2}{K(t)} = 1 + \frac{Y(t)^2}{K(t)}$$

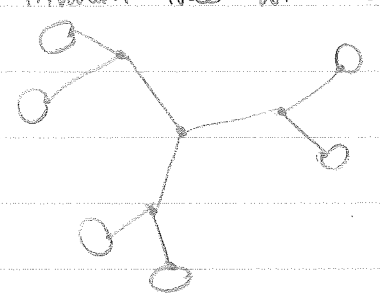
All preimages of 0 should be deg 3

1 should be deg 2

Above ∞, $m+1$ faces of deg 1 and one face of deg $6m-(m+1) = 5m-1$.

How to find?

Draw a trivalent tree with $2m$ vertices, then add loops



(Some lengthy examples here)

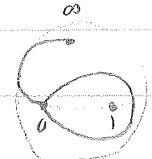
Same for abc

Take the dessin with deg 1 vertex at ∞

deg 3 vertex at 0

loops around 1

On $\mathbb{P}^1$ pictorially:



Get a Belyi fn, $f(x) = \dfrac{64x^3}{(x+9)^3(x+1)}$

Comparing $f(x)$ and $f(x)-1$ at $x = a/b$ gives an algebraic expression from which one can get "counterexamples" to ABC.