BUNTES: Three short stories about Belyi's Thm

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Ref: "Unifying themes suggested by Belyi's Thm"
— W. Goldring, '09

Thm: Let X be a (smooth projective) curve $/\mathbb{C}$.
Then X is defined over $\overline{\mathbb{Q}}$ $\iff$ There exists a map $\varphi : X \to \mathbb{P}^1$ such that $B(\varphi) \subseteq \{0, 1, \infty\}$.
A "Belyi map".

① "The case of the rising degree"

Dfn: The Belyi degree of $X/\overline{\mathbb{Q}}$ (a curve) is the minimal degree of $\varphi : X \to \mathbb{P}^1$ a Belyi map.

Q. How does the Belyi degree of $X/\overline{\mathbb{Q}}$ relate to the arithmetic of X?

Dfn: The field of moduli of $X/\overline{\mathbb{Q}}$ is the intersection over all fields $\subseteq \overline{\mathbb{Q}}$ over which X is defined.
(Similarly for a morphism $\varphi : X \to Y$)

Note: The field of moduli need not be a field of definition for X.

Given $X/\overline{\mathbb{Q}}$ with field of moduli K, we say X has good (semistable) reduction at $\mathfrak{p} \subseteq \mathcal{O}_K$ if there exists a model for X over $\mathcal{O}_{K_\mathfrak{p}}$ such that the special fibre is smooth (resp. semisimple).

For $p \in \mathbb{Z}$, say X has good (semistable) reduction at p if it does for all $\mathfrak{p} | p$, $\mathfrak{p} \subseteq \mathcal{O}_K$.

Thm [Zapponi]: If $X/\overline{\mathbb{Q}}$, then the Belyi degree of X is greater than the largest prime $p \in \mathbb{Z}$ such that X has bad semistable reduction at p.

Remark: ① The lower bound is not "sharp" because there exist E/F with good reduction everywhere, but no degree 1 map $E \longrightarrow \mathbb{P}^1$.

② $E: y^2 = x^3 + x^2 + p$

Then E has bad semistable reduction at p.
Thus the Belyi degree of $E \geqslant p$.

Thm [Beckmann]: Let $\varphi : X \longrightarrow \mathbb{P}^1$ be a Belyi map with field of moduli M for φ.
Let G be the Galois group of the Galois closure of φ.
Then $\forall p$ such that $p \nmid (\# G)$
$$\tilde{\varphi} : \tilde{X} \longrightarrow \mathbb{P}^1$$
has good reduction at p and p is unramified in M.

Remark: $\tilde{X}$ is the Galois closure $$\begin{tikzcd} \tilde{X} \arrow[r] \arrow[rr, bend right, "\tilde{\varphi}"'] & X \arrow[r, "\varphi"] & \mathbb{P}^1 \end{tikzcd}$$

Exercise: Show the thm $\Rightarrow$ X has good reduction at p.

Pf of [Zapponi]: Let $\varphi: X \longrightarrow \mathbb{P}^1$ be a Belyi map of deg n.

Let K be the field of moduli of X, M the field of moduli of φ.

Then M/K is a finite extension.

Take G as above.

Let $\mathfrak{p} \subseteq \mathcal{O}_K$ be a place of bad semistable reduction for X.

Then $\mathfrak{P} | \mathfrak{p}$, $\mathfrak{P} \subseteq \mathcal{O}_M$ is a place of bad semistable reduction for φ.

By [Beckmann], $p | (\# G)$ for $p \in \mathbb{Z}$ below $\mathfrak{p}$.

But $G \hookrightarrow S_n$.

Thus $p | n! \Rightarrow p \leq n$. $\square$

② "Finitist's Dream"

Recall that if k is a perfect field of characteristic p, then $\varphi: C_1 \longrightarrow C_2$ a morphism of curves/k is said to be tamely ramified at $P \in C$ if $e_\varphi(P)$ is not divisible by p (wildly ramified if $e_\varphi(P)$ is divisible by p).

Thm (Wild p-Belyi): For C a curve/k (char p, perfect), there exists a "wild Belyi map" $\varphi: C \longrightarrow \mathbb{P}^1$ such that $B(\varphi) = \{\infty\}$, i.e. every curve/k is birational to an étale cover of $\mathbb{A}^1$.

Ex. $\mathbb{G}_m \longrightarrow \mathbb{A}^1$, by $x \longmapsto x^p + \frac{1}{x}$

However, the tame étale fundamental group of $\mathbb{A}^1 = 0$.

Thm (Tame p-Belyi) [Saidi]: Let $p > 2$.

For $C/\overline{\mathbb{F}}_p$, there exist $\varphi : C \longrightarrow \mathbb{P}^1$ tamely ramified everywhere with $B(\varphi) \subseteq \{0, 1, \infty\}$.

Lemma [Fulton]: Let $p > 2$.

Then for C/k ($k = \bar{k}$, char p), there exists $\psi : C \longrightarrow \mathbb{P}^1$ such that $e_\psi(P) \leq 2$.

Pf of Tame p-Belyi: Take $\psi : C \longrightarrow \mathbb{P}^1$ as in [Fulton].

Then $B(\psi) \subseteq \mathbb{P}^1(\mathbb{F}_{p^m})$ for some m.

Define $f : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ by $x \mapsto x^{p^m - 1}$.

$$\begin{tikzcd} C \arrow[r, "\psi"] \arrow[rr, bend right, "\varphi"'] & \mathbb{P}^1 \arrow[r, "f"] & \mathbb{P}^1 \end{tikzcd}$$

, i.e. $\varphi = f \circ \psi$

Then φ is tamely ramified everywhere, and $B(\psi) \subseteq \{0, 1, \infty\}$. $\square$

In the $p = 2$ case, the analogue of [Fulton] is:

$\exists\, \tau : C \longrightarrow \mathbb{P}^1$ for $\mathrm{char}(k) \neq 3$ s.t. $e_\tau(P) = 1$ or 3.

③ "In the Stacks"

Observation: $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ is the moduli space of genus 0 curves with 4 (ordered) marked points.

$(\mathbb{P}^1; \alpha_1, \alpha_2, \alpha_3, \alpha_4) \longmapsto$ image of α_4 under $\alpha_1 \longmapsto 0$

$\alpha_2 \longmapsto 1$

$\alpha_3 \longmapsto \infty$

Dfn: Let $\mathcal{M}_{g,n}$ be the moduli space of genus g with n (ordered) marked points.
Let $\mathcal{M}_{g,[n]}$ —— " —— (unordered) — " —.

If n is large enough relative to g, then $\mathcal{M}_{g,n}$ will be a scheme.
However, $\mathcal{M}_{g,[n]}$ is not.

Example: $\mathcal{M}_{0,4} \simeq \mathbb{P}^1 \setminus \{0, 1, \infty\}$

Braungardt's Question: Is every $X/\overline{\mathbb{Q}}$ (smooth, projective variety) birational to a finite étale cover of some $\mathcal{M}_{g,[n]}$?

Note: There exists an étale map $\mathcal{M}_{g,n} \longrightarrow \mathcal{M}_{g,[n]}$ by forgetting the ordering of the points.

The dimension 1 case is Belyi's theorem, by

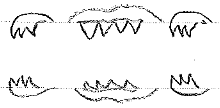
$$X \setminus \varphi^{-1}(B(\varphi)) \longrightarrow \mathbb{P}^1 \setminus \{0, 1, \infty\} \simeq \mathcal{M}_{0,4} \longrightarrow \mathcal{M}_{0,[4]}.$$

Dim 2

Among these moduli spaces only $\mathcal{M}_{1,[2]}$ and $\mathcal{M}_{0,[5]}$ are 2-dim.
We also have an étale map

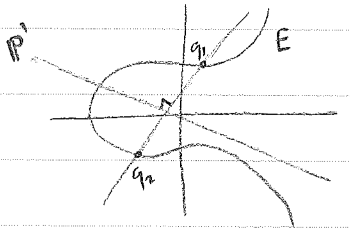
$$\mathcal{M}_{1,[2]} \longrightarrow \mathcal{M}_{0,[5]}$$

as follows ...



$\eta = (E; \{q_1, q_2\}) \in \mathcal{M}_{1,[2]}$

$\alpha(\eta) = (\mathbb{P}^1; \{r_1, r_2, r_3, r_4, r_5\})$, where the r_i come from:



$\varphi : E \longrightarrow \mathbb{P}^1$

Then

$B(\varphi) = \{r_1, r_2, r_3, r_4\}$

and $r_5 = \varphi(q_1) = \varphi(q_2)$.

So Braungardt for surfaces: Does there exist $\varphi : X \longrightarrow \mathcal{M}_{0,[5]}$ which is étale?

<u>Thm [Braungardt]</u>: For $X/\overline{\mathbb{Q}}$ an abelian surface, X is birational to an étale cover of $\mathcal{M}_{0,[5]}$.

$\frac{1}{2}$ Proof: For an abelian surface $/\overline{\mathbb{Q}}$, there exists another isogenous to it which is principally polarised.

Such surfaces come in two flavours: $E_1 \times E_2$, or $J(C)$ with $g(C) = 2$.

<u>Case 1</u>: $A = E_1 \times E_2$

Let $\varphi_i : E_i \longrightarrow \mathbb{P}^1 \setminus \{0, 1, \infty\}$ be Belyi maps.

Then $\alpha : A \xrightarrow{\varphi_1 \times \varphi_2} \mathbb{P}^1 \times \mathbb{P}^1$.

Then α restricts to a finite unramified cover $\alpha^{-1}(S) \xrightarrow{\alpha} S$ where

$S = (\mathbb{P}^1 \setminus \{0, 1, \infty\} \times \mathbb{P}^1 \setminus \{0, 1, \infty\}) \setminus \Delta$.

Note that $S \simeq \mathcal{M}_{0,[5]}$ by $(a, b) \longmapsto (\mathbb{P}^1; \{0, 1, \infty, a, b\})$

So A is birational to $\alpha^{-1}(S)$ which is an étale cover of $\mathcal{M}_{0,5}$.

<u>Case 2</u>: $A = J(C)$.

Use $\varphi: C \longrightarrow \mathbb{P}^1$ and a relation between A and $\mathrm{Sym}^2(C)$. $\square$