BUNTES

Jim LaDouce
Apr 27, 2018

Physics: Let M be a manifold with a metric g
We call the pair (M, g) a "spacetime manifold".

$\mathcal{E}$ = "space of fields" = either
- $C^\infty(M)$
- $\{$sections of $\begin{smallmatrix} E \\ \downarrow \\ M \end{smallmatrix}\}$
- $\{$connections ...$\}$

or similar

$$S(\phi) = \int_M \mathcal{L}(\phi), \quad \phi \in \mathcal{E}, \quad \mathcal{L} = \text{Lagrangian}$$

"Physically realisable states" ~ fields ϕ that minimise $S(\phi)$.

W = Superpotential: This is a term in $\mathcal{L}$ that satisfies some special symmetries

eg. could also have $S(\phi_1, \phi_2) = \int_M \mathcal{L}(\phi_1, \phi_2)$
The W might satisfy $W(\phi_1, \phi_2) = W(\phi_2, \phi_1)$.

Gauge Transformations

$G \circlearrowright E$ an action s.t. each fibre $E_x = p^{-1}(x)$ is a representation of G.
$\downarrow p$
M

A gauge is a section $s(x)$ of $\begin{smallmatrix} E \\ \downarrow \\ M \end{smallmatrix}$.

A gauge transformation is a map $g: M \longrightarrow G$ s.t. $g(x) \cdot s(x)$ is another section, call G the gauge group.

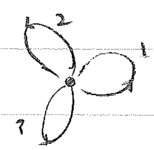
The important gauge transformations are the ones that fix the set of physically realisable states (i.e. fixes the subset of Σ that minimise S).

Let's now study the relationship between quivers and dessins.

Ex. $\mathcal{N}=4$ SYM (Supersymmetric Yang Mills)

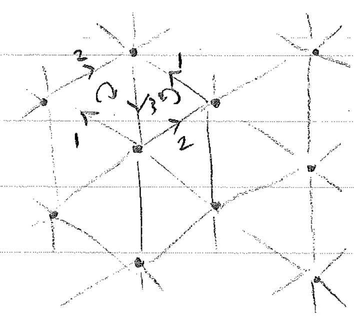
(Gauge symmetries given by some product of $SU(N)$)

Quiver:



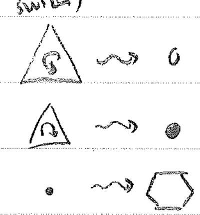
- nodes ~ factors of the gauge group (eg. here $G = SU(N)$)
- arrows ~ fields (eg. here 3 fields X_1, X_2, X_3)

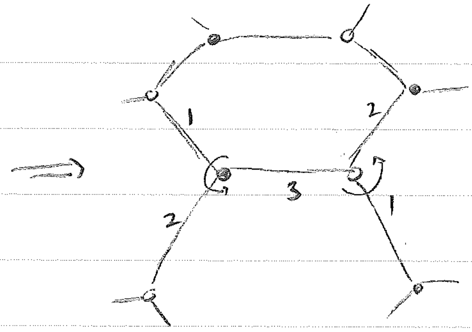
Periodic quiver: (tiling of the plane)



Take the dual (i.e. faces and nodes switch)

In particular





This is a <u>Dimer model</u>

Relating the Dimer model back to the physics

 ~ factors of the gauge group

——— ~ fields

$\bullet, \circ$ ~ terms in W

ie. · 1 distinct face $\Rightarrow$ 1 factor in the gauge group $\Rightarrow$ $G = SU(N)$

· 3 distinct edges $\Rightarrow$ 3 fields X_1, X_2, X_3

· To recover W, consider the permutation arising from reading the edges around the vertices counterclockwise

$$\bullet \sim (123) = \sigma_B \sim \text{positive terms in } W$$

$$\circ \sim (123) = \sigma_W \sim \text{negative terms in } W$$

$$\sigma_\infty = (\sigma_B \sigma_W)^{-1} = (123)$$

σ_i gives an term for each cycle in σ_i.

Each cycle in σ_B gives a product of fields indexed by the cycle,

eg. in this example σ_B gives $X_1 X_2 X_3$

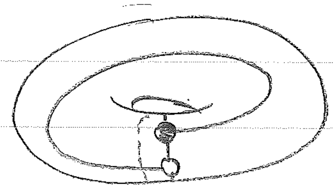
Each cycle in σ_W^{-1} gives a product of fields indexed by the cycle

eg. in this example σ_W gives $X_1 X_3 X_2$

$$\Rightarrow W = \mathrm{Tr}\big((\text{sum of } \sigma_B \text{ terms}) - (\text{sum of } \sigma_W \text{ terms})\big)$$
$$= \mathrm{Tr}(X_1 X_2 X_3 - X_1 X_3 X_2)$$

$$\mathrm{Aut}(\{\sigma_B, \sigma_W, \sigma_\infty\}) = \{\gamma \in S_3 \mid \gamma \sigma_i \gamma^{-1} = \sigma_i\}$$
$$= \{1, (123), (132)\}$$
$$= \mathbb{Z}/3\mathbb{Z}$$

$\Rightarrow$ Fundamental domain of Dimer gives



This corresponds to the Belyi pair (Σ, β) where

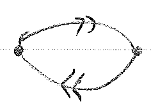
$\Sigma : y^2 = x^3 + 1, \quad \beta : \Sigma \longrightarrow \mathbb{P}^1$

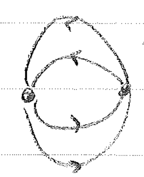
$(x,y) \mapsto \frac{y+1}{2}$

$\mathrm{Aut}(\Sigma, \beta) \cong \mathrm{Aut}\left(\{\sigma_B, \sigma_W, \sigma_\infty\}\right)$

$\mathrm{Aut}(\Sigma, \beta)$ is generated by $(x, y) \longmapsto (\omega x, y)$ where $\omega^3 = 1$.

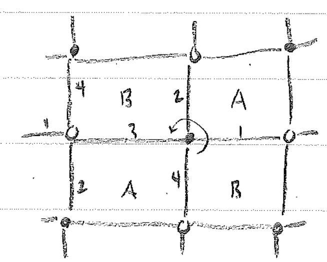
Another example



(Note that this is just notation for )

$\Rightarrow$ 4 fields, 2 factors of G (i.e. $G = SU(N) \times SU(N)$).

$\rightsquigarrow$ Dimer



$\sigma_B = (1234)$

$\sigma_W = (1234)$

$\sigma_\infty = (13)(24)$

$\Rightarrow W = \mathrm{Tr}\left(X_1 X_2 X_3 X_4 - X_1 X_4 X_3 X_2\right)$

In this case the Belyi pair is $\Sigma : y^2 = x(x-1)(x-\frac{1}{2})$

$\beta = \frac{x^2}{2x-1}$

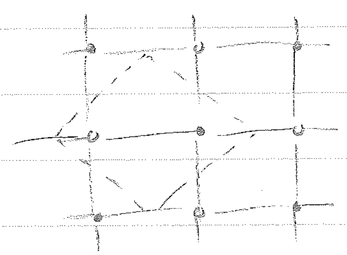
$$\mathrm{Aut}(\{\sigma_B, \sigma_w, \sigma_\infty\}) = \langle (1234) \rangle \cong \mathbb{Z}/4\mathbb{Z}$$

$$\phi_\pm : (x,y) \longmapsto \left(\frac{x}{2x-1}, \frac{\pm i y}{(2x-1)^2} \right)$$

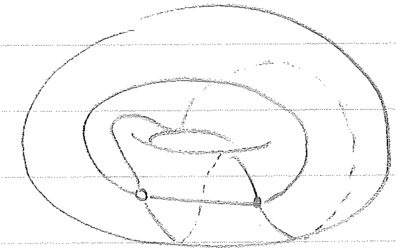
$$\phi_+^2 = \phi_-^2 : (x,y) \longmapsto (x, -y)$$
$$\phi_+^3 = \phi_+^{-1} = \phi_-$$
$$\phi_+^4 = \mathrm{Id} \quad \Rightarrow \quad \mathrm{Aut}(\Sigma, \beta) \cong \mathbb{Z}/4\mathbb{Z}$$

$$\beta^{-1}(0) = \{(0,0)\}$$
$$\beta^{-1}(1) = \{(1,0)\}$$
$$\beta^{-1}(\infty) \quad \{(\tfrac{1}{2}, 0), (\infty, \infty)\}$$

On the Dimer we have the following fundamental domain:

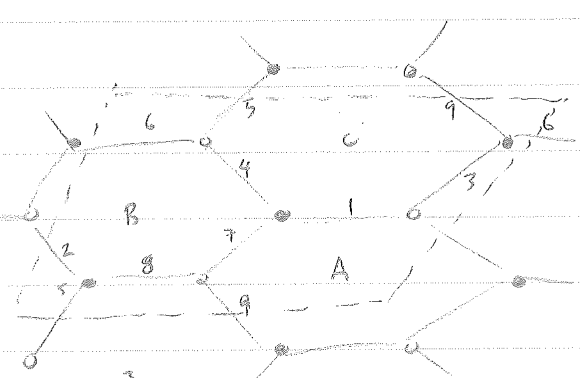


$\rightsquigarrow$



Final Example

Let's jump straight to the Dimer, with fundamental domain in the dotted line



- 9 fields
- 3 factors in the gauge group, $G = SU(N)^3$

$$\sigma_B = (147)(258)(369)$$
$$\sigma_W = (123)(456)(789)$$
$$\sigma_\infty = (195)(276)(384)$$

So $W = \mathrm{Tr}\left(\sum_{i,j,k} X^i_{12} X^j_{23} X^k_{31} \varepsilon_{ijk} \right)$

where
$$\varepsilon_{ijk} = \begin{cases} 1, & \text{if } (i,j,k) = (1,2,3),\ (2,3,1) \text{ or } (3,1,2) \\ -1, & \text{if } (i,j,k) = (3,2,1),\ (1,3,2) \text{ or } (2,1,3) \\ 0, & \text{if } i=j,\ j=k \text{ or } i=k \end{cases}$$

X^i_{12} acts on the ith field by $(N, \bar{N}, 1)$ $\quad N$ = canonical rep
$\bar{N}$ = anticanonical rep
1 = trivial

$$\mathrm{Aut}(\{\sigma_B, \sigma_W, \sigma_\infty\}) \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$$

Now for the Belyi pair, Σ = projective closure of $F = \{(x,y) \mid x^3+y^3=1\}$
$\beta(x,y) = x^3$

$$\gamma_1(x,y) = (\omega_1 x, y)$$
$$\gamma_2(x,y) = (x, \omega_2 y), \quad \omega_i^3 = 1$$