BUNTES

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Riemann Surfaces I

① Definitions

Dfn: A topological surface is a Hausdorff space X which has a collection of charts

$$\{\varphi_i : U_i \xrightarrow{\sim} \varphi_i(U_i) \underset{\text{open}}{\subseteq} \mathbb{C}\}_{i\in I}$$

such that

$$X = \bigcup_{i\in I} U_i$$

We call X a Riemann surface if the transition functions $\varphi_i \circ \varphi_j^{-1}$ are holomorphic.

② Examples

(0) Open subsets of $\mathbb{C}$, eg.

$\mathbb{C}$

$\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$

$\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$

(1) $\hat{\mathbb{C}}$ = Riemann sphere $= \mathbb{C} \cup \{\infty\}$

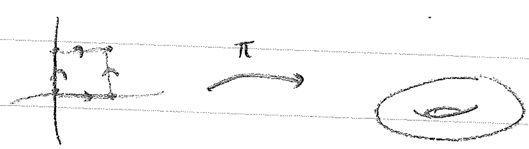
A basis of nbhds of ∞ is given by $\{z \in \mathbb{C} \mid |z| > R\} \cup \{\infty\}$

(2) $\mathbb{P}^1(\mathbb{C}) = \{ [z_0 : z_1] \mid z_0 \text{ and } z_1 \text{ not both } 0 \}$

$$U_0 = \{ [z_0 : z_1] \mid z_0 \neq 0 \} \longrightarrow \mathbb{C}$$
$$[z_0 : z_1] \longmapsto z_1/z_0$$
$$U_1 = \{ [z_0 : z_1] \mid z_1 \neq 0 \} \longrightarrow \mathbb{C}$$
$$[z_0 : z_1] \longmapsto z_0/z_1$$

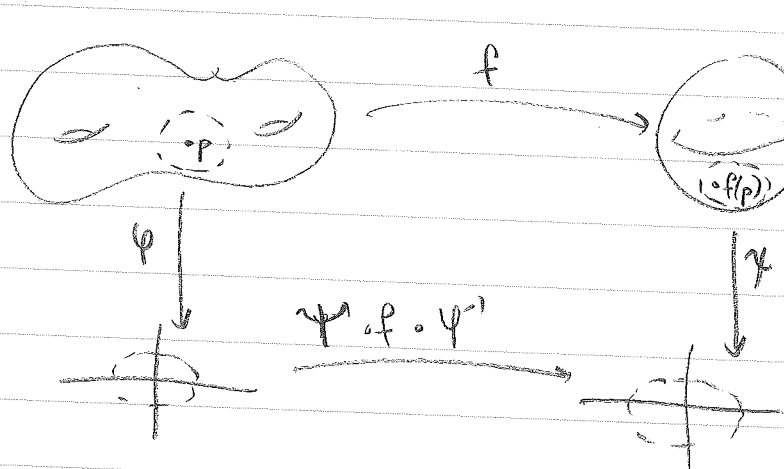
(3) Let $\Lambda = \mathbb{Z} \oplus \mathbb{Z}i \subseteq \mathbb{C}$

Then $X = \mathbb{C}/\Lambda$ is a Riemann surface.



③ Morphisms

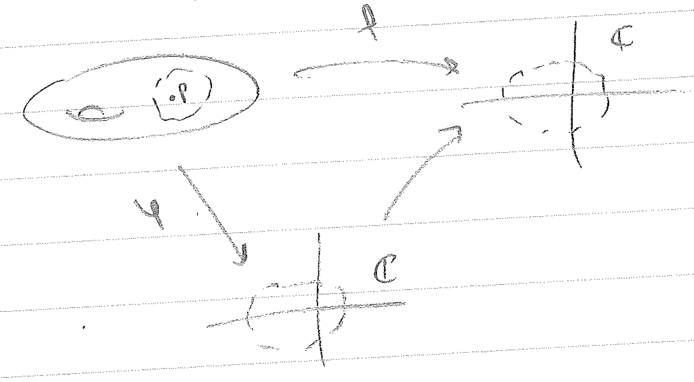
<u>Dfn</u>: A <u>morphism of Riemann surfaces</u> is a continuous map
$$f : S \longrightarrow S'$$
such that



$\psi \circ f \circ \varphi^{-1}$ is holomorphic for all such charts.

We call a morphism $f: S \longrightarrow \mathbb{C}$ a <u>holomorphic function on S</u>,

(ie.



$f \circ \varphi^{-1}$ is holomorphic)

We say $f: S \longrightarrow \mathbb{C}$ is <u>meromorphic</u> if $f \circ \varphi^{-1}$ is meromorphic.

<u>Exercise</u>: The set of meromorphic functions on a Riemann surface form a field.

We denote this field by $\mathcal{M}(S)$

<u>Prop 1.26</u>: $\mathcal{M}(\hat{\mathbb{C}}) = \mathbb{C}(z)$

<u>Pf</u>: Let $f: \hat{\mathbb{C}} \to \mathbb{C}$ be meromorphic.
Then the number of poles of f is finite, say at $a_1, \dots, a_n$.

So, locally at a_i we can write

$$f(z) = \sum_{j=1}^{j_i} \frac{\lambda_{j,i}}{(z-a_i)^j} + h_i(z)$$

with h_i holomorphic.

Then

$$f(z) - \sum_{i=1}^{n} \sum_{j=1}^{j_i} \frac{\lambda_{j,i}}{(z-a_i)^j}$$

is holomorphic everywhere.

By Liouville, this is constant. $\square$

We say S, S' are isomorphic if $\exists f: S \to S'$, $g: S' \to S$ morphisms s.t. $f \circ g = id_{S'}$, $g \circ f = id_S$.

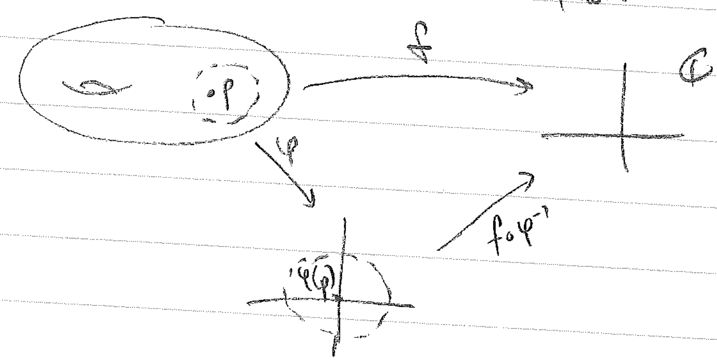
Exercise: Show $\tilde{\mathbb{C}} \simeq \mathbb{P}^1(\mathbb{C})$

Remarks: (a) $\mathbb{C} \not\simeq \mathbb{D}$ (Liouville)

(b) If S, S' are connected, compact Riemann surfaces, then any nonconstant morphism $f: S \to S'$ is surjective. (Nonconstant holomorphic maps are open).

(4) Ramification

Dfn: The order of vanishing at $P \in S$ of a holomorphic function on S $(f: S \to \mathbb{C})$ is defined as follows

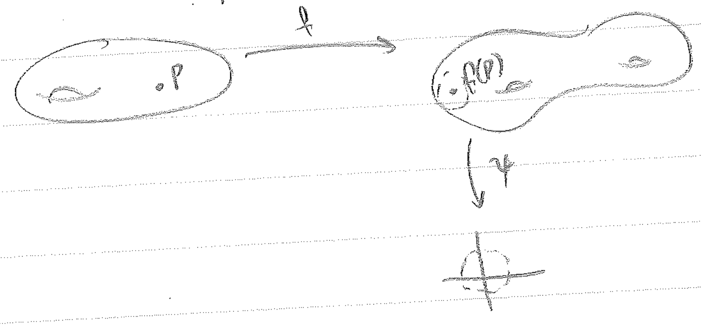


for φ a chart centred at P, write

$$f \circ \varphi^{-1}(z) = a_n z^n + a_{n+1} z^{n+1} + \cdots \qquad a_n \neq 0.$$

Then $\mathrm{ord}_P(f) := n$.

More generally, for $f: S \longrightarrow S'$ we can define $m_P(f)$ (multiplicity of f at P) by using



and setting $m_P(f) := \mathrm{ord}_P(\psi \circ f)$

If $m_P(f) \geq 2$, call P a branch point of f and call f ramified at P.

Example: $f: \mathbb{C} \longrightarrow \mathbb{C}$, $f(z) = z^2$

The chart $\varphi_a(z) = z - a$ is centred at $a \in \mathbb{C}$.
Then to compute $m_a(f)$, we compute

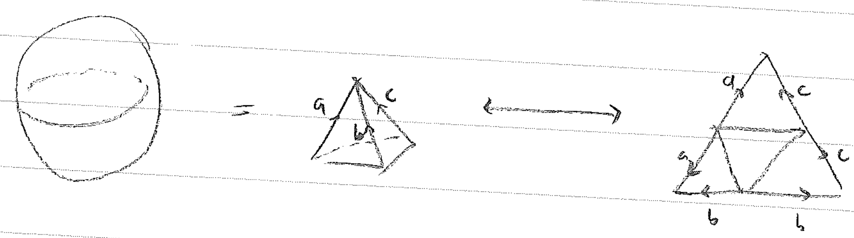
$$f \circ \varphi_a^{-1}(z) = a^2 + 2az + z^2.$$

Hence

$$\mathrm{ord}_a(f) = \begin{cases} 0, & \text{if } a \neq 0 \\ 2, & \text{if } a = 0. \end{cases}$$

(5) Genus

Theorem [Radó]: Any orientable compact surface can be triangulated.



Fact: Riemann surfaces are orientable.

(Given $\varphi \circ \psi^{-1}(x,y) = (u(x,y), v(x,y))$

$$\left|\frac{\partial(\varphi\circ\psi^{-1})}{\partial(x,y)}\right| = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} u_x & -v_x \\ v_x & u_x \end{vmatrix} > 0$$)

Given such an oriented polygon coming from a Riemann surface, we can associate a word w to it from travelling around the perimeter.

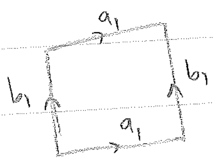
Example: For the sphere, $w = a^{-1}ab^{-1}bc^{-1}c$

Fact: Every such word can be normalised without changing the corresponding Riemann surface

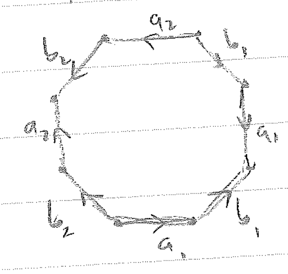
$$w = \begin{cases} w_0 = aa^{-1} \\ w_g = a_1b_1a_1^{-1}b_1^{-1} \cdots a_gb_ga_g^{-1}b_g^{-1}. \end{cases}$$

The (uniquely determined) g is the genus of the surface.

Example: $w_1 = a_1 b_1 a_1^{-1} b_1^{-1}$



$w_2 = a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1}$



Theorem: $\chi(S) = v - e + f$

$$= 2 - 2g(S)$$