Riemann Surfaces II

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Riemann Hurwitz Formula

Ex 1 (Unimportant): The genus of a Riemann surface is invariant under triangulation.

In particular, there are (at least) two distinct ways of thinking about Riemann surface genus:

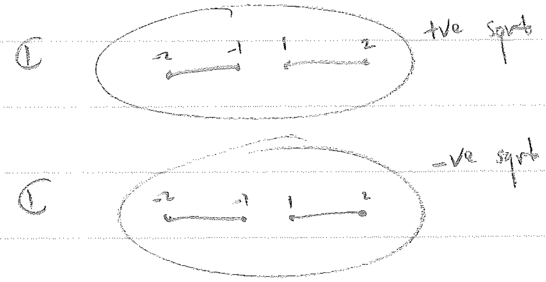
1. Given a triangulation, we can compute the Euler characteristic χ. Then g satisfies $\chi = 2 - 2g$.

2. The genus = dimension of space of holomorphic differentials.

Goal: Given a Riemann surface R, compute the genus.

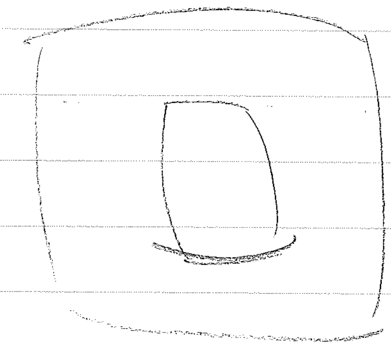
e.g. $\{(x,y) \mid y^2 = (x+1)(x-1)(x+2)(x-2)\}$.

If we consider "$y = \pm\sqrt{(x-1)(x+1)(x-2)(x+2)}$" we get 2 values of y for each x, except at the points $x = -2, -1, 1, 2$.
We have some branch cuts



Going into a branch on the +ve $\mathbb{C}$ one comes out on the -ve $\mathbb{C}$. This tells us how to glue.

In this instance, it gives us:



$\rightsquigarrow$ genus $= 1$

This was a very ad hoc approach, but it suggests the following philosophy:

- Examine the values where there were not enough preimages when we plugged in a value of x
- <u>Idea</u>: Project to x, understand # of preimages.

Now consider $P(x,y) = y^n + P_{n-1}(x)y^{n-1} + \dots + P_1(x)y + P_0(x)$
an irreducible polynomial, smooth.

Let
$$R = \{(x,y) \mid P(x,y) = 0\}.$$

Fix $x_0 \in \mathbb{P}^1\mathbb{C}$ (thinking of this as $\mathbb{C} \cup \{\infty\}$).
We expect n y-values giving points $(x_0, y) \in R$.

There are only finitely many x-values for which there are fewer since these correspond to roots of the discriminant of P.

Dfn: We say x_0 is a <u>branch point</u> if there are fewer than n distinct y-values above x.

The <u>total branching index</u> is

$$b := \sum_{x \in \mathbb{P}^1\mathbb{C}} \left(\deg(\pi) - \# \pi^{-1}(x) \right)$$

where in the example above, $\pi : R \longrightarrow \mathbb{P}^1\mathbb{C}$
$(x,y) \mapsto x$.

We can then compute $\chi(R) = (\deg \pi)\, \chi(\mathbb{P}^1\mathbb{C}) - b$.

<u>Lemma</u>: Locally given some choice of coordinates, a nonconstant morphism of Riemann Surfaces $f : R \longrightarrow S$ is given by $w \mapsto w^m$.

More precisely, given $r \in R$ and $s = f(r)$, pick $V_s \ni s$ a small nbhd and identification $\psi : V_s \to \mathbb{D}$ which sends $s \mapsto 0$.
Similarly find U_r for r.

$$\begin{tikzcd} r \in U_r \arrow[r, "f"] \arrow[d, "\phi"'] & V_s \ni s \arrow[d, "\psi"] \\ \mathbb{D} \arrow[r] & \mathbb{D} \\ w \arrow[r, mapsto] & w^m \end{tikzcd}$$

Pf: In the notes. $\square$

Pf for $\chi(R)$: Triangulate R such that every face lies in some coordinate nbhd s.t. $\pi : R \to \mathbb{P}^1\mathbb{C}$ is given by $w \mapsto w^m$, similarly for every edge, and all branch points are vertices.

This ensures that every face, edge and vertex has $n = \deg(\pi)$ preimages (except branch points).

$\Rightarrow \deg(\pi) - \# \pi^{-1}(x)$ preimages.

To compute $\chi(R)$, we consider $\deg(\pi) \cdot \chi(\mathbb{P}^1\mathbb{C})$.
However, we've overcounted preimages of branch points by precisely b.
Thus $\chi(R) = \deg(\pi) \cdot \chi(\mathbb{P}^1\mathbb{C}) - b$ $\square$

Example: Given $P(x,y)$ a plane curve defined by an equation of degree d,

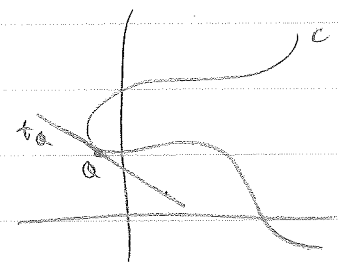
$$g = \frac{(d-1)(d-2)}{2}.$$

Consider $\mathbb{P}^2 = \{[x:y:z]\}$

$$(\mathbb{P}^2)^* = \{\text{lines in } \mathbb{P}^2\} = \{[a:b:c]\} \cong \mathbb{P}^2$$
$$aX + bY + cZ = 0$$

So we have a duality lines $\leftrightarrow$ points.

Given a curve C, at each $Q \in C$ we can compute the tangent line t_Q.



This defines a point in $(\mathbb{P}^2)^*$.

$\rightsquigarrow$ Get a <u>dual curve</u> $C^* = \{ t_Q \in (\mathbb{P}^2)^* \mid Q \in C \}$.

<u>Claim:</u> $\deg C^* = d(d-1)$

Consider $\pi : R = \{ (x,y) \mid P(x,y) = 0 \} \longrightarrow \mathbb{P}^1\mathbb{C}$

$$(x,y) \longmapsto x$$

We want to compute b.

The claim above can be stated as: if we fix a point $Q \in C$, then there are $d(d-1)$ lines through Q which are tangent to C.

<u>Note:</u> Projecting to x-coordinate $\leftrightsquigarrow$ Family of lines through point at ∞ $\leftrightsquigarrow$ * line in $(\mathbb{P}^2)^*$.

So our question about π is reduced to: how many points on C^* does this line intersect? (up to mult.)

By Bezout's Thm, $\deg C^*$.

<u>Pf (Matt Emerton):</u> Consider a point on C in $\mathbb{P}^2$ such that no tangent to the curve at ∞ passes through it, move pt to origin. If we write $P(x,y) = f_d + f_{d-1} + \dots + f_0$, then this condition is equivalent to $(f_d, f_{d-1}) = 1$.

Suppose they share a linear factor:

$$0 = (f_d)_x x + (f_d)_y y + f_{d-1}$$

Then this defines a line through the origin (because this gives the equation of an asymptote, this is a contradiction).

$$f_d + f_{d-1} + \cdots + f_0 = 0$$
$$d f_d + (d-1) f_{d-1} + \cdots + f_1 = 0$$

$$\Rightarrow \begin{cases} f_d + f_{d-1} + \cdots + f_0 = 0 \\ f_{d-1} + 2f_{d-2} + \cdots + (d-1) f_1 = 0 \end{cases}$$

Now these have $d(d-1)$ common solutions.

$\Rightarrow C^*$ has degree $d(d-1)$

$\Rightarrow b = d(d-1)$

Then $\chi(R) = 2\deg(\pi) - d(d-1)$
$= 2d - d(d-1)$ $\Rightarrow g = \frac{(d-1)(d-2)}{2}$ □

A 3-fold equivalence of categories

① Analysis: Compact, connected Riemann surfaces

② Algebra: Field extensions $K/\mathbb{C}$ where K is finitely generated of transcendence degree 1 over $\mathbb{C}$

③ Geometry: Complete nonsingular irreducible algebraic curves in $\mathbb{P}^n$.

③ $\rightsquigarrow$ ②: Take field of rational functions $\frac{p(x)}{q(x)}$, $\deg p = \deg q$,
$p, q : C \to \mathbb{C} \cup \{\infty\}$.

③ $\rightsquigarrow$ ①: Take complex structure induced by $\mathbb{P}^n$.

① $\rightsquigarrow$ ②: Associated field of meromorphic functions on X

① $\rightsquigarrow$ ③: We take a holomorphic embedding into $\mathbb{P}^n$, then use Riemann-Roch to get the algebraic relations

② $\rightsquigarrow$ ①, ③: $K/\mathbb{C}$, consider valuation rings R s.t. $K \supseteq R \supseteq \mathbb{C}$.

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eg. • $g=0$  $\mathbb{P}^1\mathbb{C}$, $\mathbb{C}\cup\{\infty\}$, $\mathbb{C}(t)$

• $g=1$ elliptic curves, $f(x,y,z)$ plane cubic

$$\mathbb{C}(x,y)/(y^2 - f(x))$$

$$\mathbb{C}/\Lambda \longleftrightarrow E \subseteq \mathbb{P}^2$$

$$z \longmapsto [z : \wp(z) ; \wp'(z)]$$

$$\int_{(x_0,y_0)}^{(x,y)} \frac{dx}{y} \longmapsfrom (x,y)$$

Riemann-Hurwitz (in general)

There is nothing that doesn't generalise from $\pi: R \to \mathbb{P}^1\mathbb{C}$ to $f: R \to S$, regarding either the statement or the proof.

Given $f: R \to S$ a nonconstant morphism of Riemann surfaces, we have

$$\chi(R) = \deg(f) \cdot \chi(S) - \left(\sum_{x \in S} \left(\deg(f) - \# f^{-1}(x)\right)\right)$$

Corollary: There are no nonconstant morphisms from a sphere to a surface of genus > 0.

Pf: $f: \mathbb{P}^1\mathbb{C} \to S$,

$$\chi(\mathbb{P}^1\mathbb{C}) = \underbrace{(\deg f)}_{+ve} \underbrace{\chi(S)}_{-ve} - \underbrace{b}_{+ve}$$

LHS $= 2$

RHS < 0. Contradiction. $\square$

Exercises: ① $x^n + y^n = z^n$ is not solvable in non-constant polynomials for $n > 2$

② Let $E = \mathbb{C}/_{\mathbb{Z} + \mathbb{Z}i}$

Multiplication by i rotates and we set $x \sim xi$.

Then $E/\sim$ is a Riemann surface.

Let $f: E \to E/\sim$, compute the branch pts of order 2, order 4.