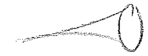
BUGLES (BU Geometry Learning Expository Seminar)



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Riemann Surfaces and Discrete Groups

<u>Plan</u>:
- Uniformisation
- Fuchsian Groups
- Automorphisms of Riemann surfaces

<u>Facts</u>: <u>Prop 1.27</u>:
- $\mathrm{Aut}(\hat{\mathbb{C}}) = \{ z \longmapsto \frac{az+b}{cz+d} \}$
- $\mathrm{Aut}(\mathbb{C}) = \{ z \longmapsto az+b \}$
- $\mathrm{Aut}(\mathbb{H}) = \{ z \longmapsto \frac{az+b}{cz+d} \mid a,b,c,d \in \mathbb{R} \}$

<u>Thm 1.69</u>:
- Σ has a universal cover, $\tilde{\Sigma}$, and $\pi_1(\tilde{\Sigma}) = 1$
- $\tilde{\Sigma} \to \Sigma$ holomorphic
- $\Sigma = \tilde{\Sigma}/G$, $G = \pi_1(\Sigma)$

G acts freely and properly discontinuously

I. Uniformisation

<u>Thm (Uniformization)</u>: The only simply connected Riemann surfaces (up to isomorphism) are $\hat{\mathbb{C}}$, $\mathbb{C}$, $\mathbb{H}$.

Combining with 1.69, we see that all RS's are a quotient of one of these three.

Thm: Let Σ be a Riemann surface.

Then
- $g=0 \Rightarrow \Sigma \cong \hat{\mathbb{C}}$
- $g=1 \Rightarrow \Sigma \cong \mathbb{C}/\Lambda$
- $g \geq 2 \Rightarrow \Sigma \cong \mathbb{H}/K$

Pf:
- $g=0$: Uniformization
- $g \geq 1$: $\hat{\mathbb{C}}$ can't be a cover (by RH)

  $g=1$: $\pi_1(\Sigma) = \mathbb{Z} \oplus \mathbb{Z}$ abelian

Lemma: No subgroup of $\mathrm{Aut}(\mathbb{H})$ isomorphic to $\mathbb{Z}\oplus\mathbb{Z}$ acting freely and prop. disc.

So $\tilde{\Sigma} = \mathbb{C}$, $z \mapsto az+b$, free if $a=1$, $z \mapsto z+\lambda_1$, $z \mapsto z+\lambda_2$

- $g \geq 2$: $\pi_1(\Sigma)$ not abelian, but $z \mapsto z+\lambda_1$ is abelian!

  $\Sigma = \mathbb{H}/K$, $K \subseteq PSL(2,\mathbb{R})$. $\square$

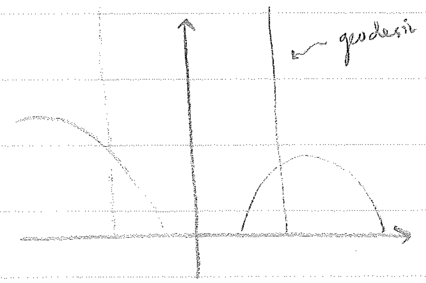
Goal: Understand Σ as $\tilde{\Sigma}/G$.

Fuchsian Groups

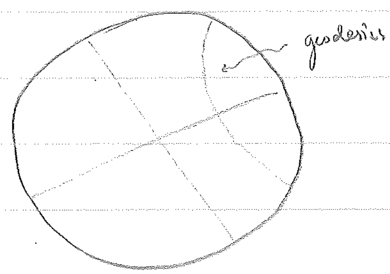
$(g \geq 2)$

$$\mathrm{Aut}(\mathbb{H}) = PSL(2,\mathbb{R}) = \mathrm{Isom}^+\left(\mathbb{H}, \frac{|dz|^2}{\mathrm{Im}\, z}\right).$$

Hyperbolic $\mathbb{H}$, $\mathbb{D}$



$PSL(2,\mathbb{R})$ acts transitively on geodesics



Dfn: A Fuchsian group is a discrete subgroup of $PSL(2,\mathbb{R})$.

Remark: (Proof in the book) Even if Γ doesn't act freely,

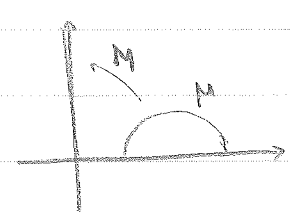
$$\mathbb{H} \longrightarrow \mathbb{H}/\Gamma$$

is still a covering map.

$\mathbb{H}/\Gamma$ has the structure of a Riemann surface.

Reflection in $\mathbb{H}$

Say μ is a geodesic.



There is $M \in PSL(2,\mathbb{R})$ such that

$M(\mu)$ = imaginary axis

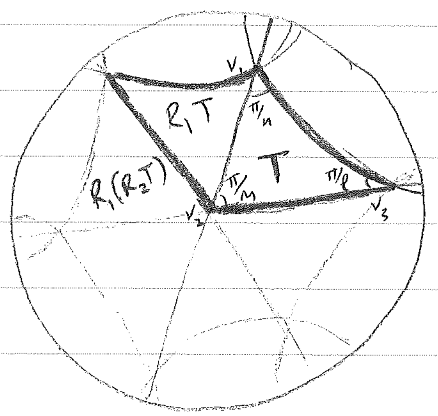
Let $R = -\bar{z}$ be reflection over the imaginary axis

Then $R_\mu = M^{-1} \circ R \circ M$ is a reflection over μ.

$$R_\mu = \frac{a\bar{z}+b}{c\bar{z}+d} \notin PSL(2, \mathbb{R})$$

Triangle groups

Given $n, m, l \in \mathbb{Z} \cup \{\infty\}$ there exists a hyperbolic triangle with angles $\pi/n, \pi/m, \pi/l$ if $\frac{1}{n} + \frac{1}{m} + \frac{1}{l} < 1$ with area $\pi\left(1 - \frac{1}{n} - \frac{1}{m} - \frac{1}{l}\right)$



Let R_1 = reflection over $\overline{v_1 v_2}$

R_2 = reflection over $\overline{v_2 v_3}$

R_3 = reflection over $\overline{v_3 v_1}$

Problem: $R_i \notin PSL(2, \mathbb{R})$

Soln: $\left.\begin{aligned} x_1 &= R_3 R_1 \\ x_2 &= R_1 R_2 \\ x_3 &= R_2 R_3 \end{aligned}\right\} \in PSL(2, \mathbb{R})$

New fundamental domain for the x_i - action is the quarilateral labelled above.

So this defines a Fuchsian group with presentation

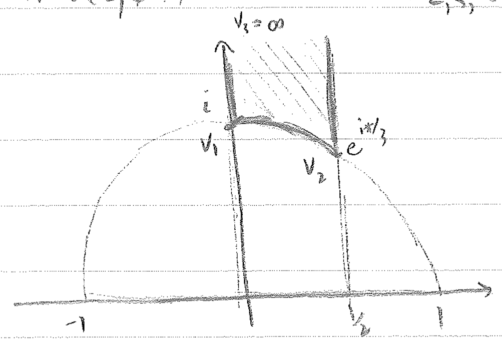
$$\langle x_1, x_2, x_3 \mid x_1^n = x_2^m = x_3^l = x_1 x_2 x_3 = 1 \rangle.$$

Note that n, m, l can equal ∞.

Dfn: Let $\Gamma_{n,m,\ell}$ be the triangle group with signature $\left(\frac{1}{n}, \frac{1}{m}, \frac{1}{\ell}\right)$.

Remark:
- $\frac{1}{n} + \frac{1}{m} + \frac{1}{\ell} = 1$ still works on $\mathbb{C}$
- $\frac{1}{n} + \frac{1}{m} + \frac{1}{\ell} > 1$ still works on $\hat{\mathbb{C}}$

Examples ($PSL(2,\mathbb{Z})$) Consider $\Gamma_{2,3,\infty}$ - angles $\pi/2, \pi/3, 0$



$$\left.\begin{aligned} R_1 &= 1/\bar{z} \\ R_2 &= 1 - \bar{z} \\ R_3 &= -\bar{z} \end{aligned}\right\} \rightsquigarrow \left\{\begin{aligned} x_1 &= -1/z \\ x_2 &= \frac{1}{1-z} \\ x_3 &= z+1 \end{aligned}\right.$$

Thm: $\Gamma_{2,3,\infty} \cong PSL(2,\mathbb{Z})$

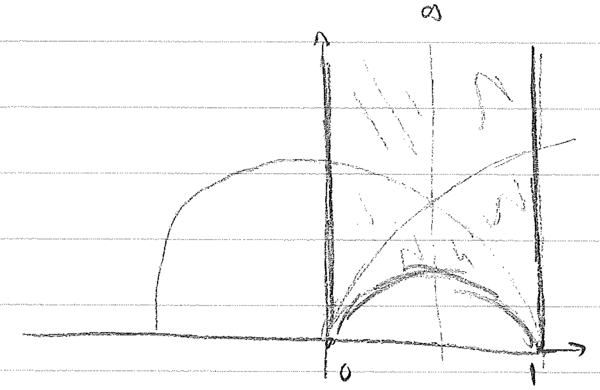
Observation: $\Gamma_1 \subseteq \Gamma_2$, T is a fundamental domain of Γ_2.
Then if $\gamma_1, \dots, \gamma_n \in \Gamma_2$ are representatives of Γ_2/Γ_1, then
$\bigcup \gamma_i(T)$ is a fundamental domain for Γ_1.

Example ($\Gamma(2)$) $\Gamma(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z}) \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv I \pmod{2} \right\}$

$[PSL(2,\mathbb{Z}) : \Gamma(2)] = 6$

Representatives of $PSL(2,\mathbb{Z})/\Gamma(2)$ are $x_1 = Id$, $x_2 = \frac{-1}{z-1}$, $x_3 = \frac{z-1}{z}$,
$x_4 = z-1$, $x_5 = \frac{-z}{z-1}$, $x_6 = -1/z$.

Translating the fundamental domain of $PSL(2,\mathbb{Z})$ we get a new fundamental domain



So angles are 0!

$\Rightarrow$ This corresponds to $\Gamma_{\infty,\infty,\infty}$ i.e.

$$(x_1, x_2, x_3 \mid x_1x_2x_3 = 1) = \langle x_1, x_2 \rangle = \pi_1(\mathbb{P}^1\mathbb{C} \setminus \{0,1,\infty\}).$$

Automorphisms of RS's ($g \geq 2$)

Prop: $S_1 = \mathbb{H}/\Gamma_1$, $S_2 = \mathbb{H}/\Gamma_2$, then $S_1 \overset{\phi}{\cong} S_2 \iff \Gamma_1 = T \circ \Gamma_2 \circ T^{-1}$, $T \in PSL(2,\mathbb{R})$

Pf: ($\Leftarrow$) Define $\phi : S_1 \xrightarrow{\sim} S_2$ by $\phi([z]_1) = [T(z)]_2$

($\Rightarrow$) Take a lift

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{\tilde{\phi}} & \mathbb{H} \\ \downarrow & & \downarrow \\ S_1 & \xrightarrow{\phi} & S_2 \end{array}$$

$T = \tilde{\phi}$

□

<u>Prop</u>: Γ Fuchsian gp, acts freely.
Then $\mathrm{Aut}(\mathbb{H}/\Gamma) = N(\Gamma)/\Gamma$

<u>Pf</u>: Previous prop set $\Gamma_1 = \Gamma_2$, get
$$N(\Gamma) \longrightarrow \mathrm{Aut}(\mathbb{H}/\Gamma).$$
Kernel is Γ. $\square$

<u>Cor</u>: Σ is a Riemann surface with $g \geq 2$. Then
$$|\mathrm{Aut}(\Sigma)| < \infty$$

<u>Pf</u>:
$$\begin{tikzcd}
\mathbb{H} \arrow[d, "\phi_2"'] \arrow[dr, "\phi_1"] & \\
\mathbb{H}/\Gamma \arrow[r, "f"'] & \mathbb{H}/N(\Gamma) = S/\mathrm{Aut}(S)
\end{tikzcd}$$

Since ϕ_1, ϕ_2 holomorphic, so is f.
$$\deg f = \# N(\Gamma)/\Gamma.$$
Since f holomorphic and these RS's are compact, $\deg f < \infty$. $\square$

Say Σ a Riemann surface with $g \geq 2$, $G \subseteq \mathrm{Aut}(\Sigma)$.
Let $\bar{g}$ be the genus of Σ/G.

$$2g - 2 = \# G \cdot (2\bar{g} - 2) + \sum_{p \in \Sigma} (\# \mathrm{Stab}(p) - 1)$$

Exercises

① Σ, $g \geq 2$; then $\# \mathrm{Aut}(\Sigma) \leq 84(g-1)$

Hint: Cases

② Consider

$$1 \longrightarrow \Gamma(n) \longrightarrow PSL(2,\mathbb{Z}) \longrightarrow PSL(2,\mathbb{Z}/_n\mathbb{Z}) \longrightarrow 1$$

and compute the genus of $\mathbb{H}/\Gamma(n)$.