The most excellent Jim Show

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§2.6. The moduli space of compact Riemann surfaces of genus g

$\underline{g=0}$: Uniformization $\Rightarrow$ $\mathcal{M}_0 = \{*\}$
(just $\mathbb{P}^1\mathbb{C}$)

$\underline{g=1}$: Uniformization $\Rightarrow$ Each RS of genus 1 can be written as
$\mathbb{C}/\Lambda \cong \mathbb{C}/\mathbb{Z}\oplus\tau\mathbb{Z}$ $\quad \tau \in \mathbb{H}$
$\Lambda_\tau := \mathbb{Z}\oplus\tau\mathbb{Z}$

Prop 2.54: $\mathcal{M}_1 \cong \mathbb{H}/PSL_2(\mathbb{Z}) \cong \mathbb{C}$

Pf: An isomorphism $\mathbb{C}/\Lambda_{\tau_1} \xrightarrow{\sim} \mathbb{C}/\Lambda_{\tau_2}$ is equivalent to $T \in \mathrm{Aut}(\mathbb{C})$
$(T(z) = w\cdot z)$ s.t. $w(\mathbb{Z}\oplus\tau_1\mathbb{Z}) = \mathbb{Z}\oplus\tau_2\mathbb{Z}$
This is equivalent to $A \in GL_2(\mathbb{Z})$, $A = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$
s.t. $A\begin{pmatrix}1\\ \tau_2\end{pmatrix} = \begin{pmatrix} w \\ w\tau_1\end{pmatrix}$
$\Rightarrow \tau_1 = A\tau_2 = \frac{a\tau_2+b}{c\tau_2+d}$, $\quad A \in PSL_2(\mathbb{R})$.
$\Rightarrow A \in PSL_2(\mathbb{Z})$. $\square$

$\underline{g>1}$: $\mathcal{M}_g$ is a complex variety of $\dim_{\mathbb{C}} 3g-3$.
Uniformisation tells us that describing a RS amounts to specifying
$2g$ real 2×2 matrices $\{\gamma_i\}_{i=1}^{2g}$ s.t.
① $\det(\gamma_i) = 1$ $\Rightarrow$ each γ_i depends on 3 real parameters
$\Rightarrow$ $6g$ $\mathbb{R}$-dim
② $\prod_{i=1}^{g}[\gamma_i, \gamma_{g+i}] = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$. $\Rightarrow$ 3 relations
$\Rightarrow$ $6g-3$ $\mathbb{R}$-dim

③ For any $\gamma \in PSL_2(\mathbb{R})$, $\Gamma = \langle \gamma_i \rangle$ and $\gamma \Gamma \gamma^{-1}$ uniformize isomorphic Riemann surfaces $\Rightarrow$ $6g-6$ $\mathbb{R}$-dim
$\Rightarrow$ $3g-3$ $\mathbb{C}$-dim

§2.7. Monodromy

Let $f : S_1 \longrightarrow S$ be a degree d morphism, ramified over a finite set $y_1, \cdots, y_n \in S$.

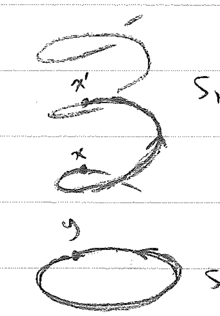
For $y \in S \setminus \{y_1, \cdots, y_n\}$, we get a group homomorphism

$$M_f : \pi_1(S \setminus \{y_1, \cdots, y_n\}) \longrightarrow \mathrm{Bij}(f^{-1}(y))$$
$$\gamma \longmapsto \sigma_\gamma^{-1}$$

where σ_γ is defined as follows

$\gamma \in \pi_1(S \setminus \{y_1, \cdots, y_n\})$ lifts to a path $\tilde{\gamma}$ from $x \in f^{-1}(y)$ to another $x' \in f^{-1}(y)$.

Set $\sigma_\gamma(x) = x'$



We set $M_f(\gamma) = \sigma_\gamma^{-1}$ so it is a group homomorphism (rather than an antihomomorphism).

If we number the points in $f^{-1}(y)$ as $\{1, \cdots, d\}$, we can think of $M_f(\pi_1) \subseteq \Sigma_d$.

Denote the image of $M_f(\pi_1)$ in Σ_d as $\mathrm{Mon}(f)$.

Monodromy and Fuchsian groups

Let $\pi : \mathbb{H}/\Gamma_1 \longrightarrow \mathbb{H}/\Gamma$ be the Fuchsian group representation of the map $f : S_1 \longrightarrow S$.

Translating the notation above:

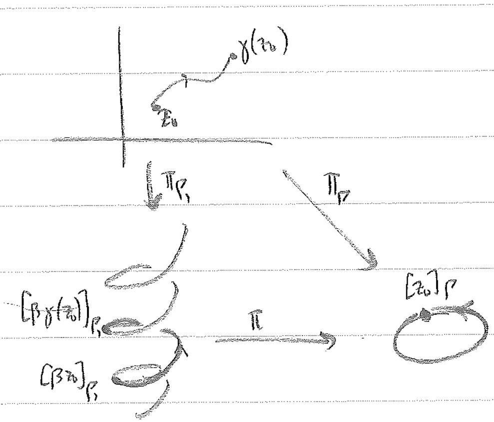
- $y = [z_0]_\Gamma$ for some $z_0 \in \mathbb{H}$
- $\pi_1(S \setminus \{y_1, \dots, y_n\}) \simeq \Gamma$
- $f^{-1}(y) = \{[\beta z_0]_{\Gamma_1}\}$ where β runs along a set of representatives of $\Gamma_1 \backslash \Gamma$.

For $\Gamma_1 \leq \Gamma$,

$$\begin{tikzcd} \mathbb{H} \arrow[d, "\pi_{\Gamma_1}"'] \arrow[dr, "\pi_\Gamma"] & \\ \mathbb{H}/\Gamma_1 \arrow[r, "\pi"] & \mathbb{H}/\Gamma \end{tikzcd}$$

$$?$$

$$S_1 \setminus f^{-1}(\{y_1, \dots, y_n\}) \xrightarrow{f} S \setminus \{y_1, \dots, y_n\}$$



$$M_f : \Gamma \longrightarrow \mathrm{Bij}(\Gamma_1 \backslash \Gamma)$$
$$\gamma \longmapsto M_f(\gamma)$$

$\Rightarrow \gamma \sim \pi_\Gamma([z_0, \gamma(z_0)])$ where $[z_0, \gamma(z_0)]$ is a path in $\mathbb{H}$. Then map this path back down to $\mathbb{H}/\Gamma_1$ to get a path with initial point $[\beta(z_0)]_{\Gamma_1}$. This is the path $\pi_{\Gamma_1}(\beta[z_0, \gamma(z_0)])$.

Cor 2.59: $M_\pi : \Gamma \longrightarrow \mathrm{Bij}(\Gamma_1 \backslash \Gamma)$ induces an isomorphism

$$\frac{\Gamma}{\bigcap_{\beta \in \Gamma_1} \beta^{-1} \Gamma_1 \beta} \xrightarrow{\simeq} \mathrm{Mon}(\pi)$$

Characterise morphisms by monodromy

Thm 2.61: Let $f_i : S_i \longrightarrow S$, $i = 1,2$ have degree d with the same branch values.
Then f_1 and f_2 have conjugate monodromies iff they are isomorphic as coverings.

Prop 2.63: For S a compact RS and $B = \{a_1, \dots, a_n\} \subseteq S$ for some $d \geq 1$ there are only finitely many pairs $(\tilde{S}, f)$ where $\tilde{S}$ is a compact RS and $f : \tilde{S} \longrightarrow S$ is a degree d morphism with branching value set B.

Pf (special case): Assume $S = \mathbb{P}^1$ and $n = 3$.

$$\Gamma = \Gamma(2) = \{ A \in PSL_2(\mathbb{Z}) \mid A \equiv I \pmod 2 \}$$

$\cong \pi_1(S \setminus \{0, 1, \infty\})$ is generated by γ_1 and γ_2

$\Rightarrow$ Any $M_f : \Gamma(2) \to \Sigma_d$ is determined by images of γ_1, γ_2 (finite number of choices). $\square$

§2.8. Galois coverings

Dfn: A covering $f : S_1 \to S_2$ is Galois (or regular, or normal) if the covering group

$$G = \mathrm{Aut}(S_1, f) = \{ h \in \mathrm{Aut}(S_1) \mid f \circ h = f \}$$

acts transitively on each fibre.

Then $S_2 \cong S_1 / G$.

Prop 2.65: $f : S_1 \longrightarrow S_2$ is Galois iff $f^* : M(S_2) \longrightarrow M(S_1)$ is a Galois extension of fields.

meromorphic functions

In this case $\mathrm{Aut}(S_1, f) = \mathrm{Gal}\left(M(S_1) / f^* M(S_2)\right)$.

Ex: Hyperelliptic covers of $\mathbb{P}^1\mathbb{C}$ given by

$$\begin{aligned} \underline{x} : S = \{(x,y) \mid y^2 = \textstyle\prod (x - a_i)\} &\longrightarrow \mathbb{P}^1\mathbb{C} \\ (x,y) &\longmapsto x \end{aligned}$$

Covering group G is of order 2 and is generated by the involution $J(x,y) = (x,-y)$.

$$\begin{tikzcd} S \arrow[d, "\pi"'] \arrow[dr, "x"] & \\ S/\langle J \rangle \arrow[r] & \mathbb{P}^1\mathbb{C} \end{tikzcd}$$

$[(x,y)]_G \longmapsto x$

Prop 2.66: A covering $f : S_1 \longrightarrow S_2$ is normal iff $\deg(f) = \# \mathrm{Mon}(f)$.

§2.9. Normalisation of coverings of $\mathbb{P}^1\mathbb{C}$

Let $f : S \longrightarrow \mathbb{P}^1\mathbb{C}$ be a cover of deg $d > 0$ with $\mathrm{Mon}(f) \subseteq \Sigma_d$.

The normalisation $\tilde{f} : \tilde{S} \longrightarrow \mathbb{P}^1\mathbb{C}$ associated to f has $\mathrm{Mon}(f) \cong \mathrm{Aut}(\tilde{S}, \tilde{f})$ and $\tilde{f}^* : M(\mathbb{P}^1\mathbb{C}) \longrightarrow M(\tilde{S})$ is the normalisation of the extension $f^* : M(\mathbb{P}^1\mathbb{C}) \longrightarrow M(S)$.

Normalisation of an extension $K \longrightarrow L$ is a Galois extension of K of lowest possible degree containing L.

Dfn 2.67: The normalisation of $f : S \to \mathbb{P}^1\mathbb{C}$, $\deg f = d > 0$, is a Galois covering $\tilde{f} : \tilde{S} \longrightarrow \mathbb{P}^1\mathbb{C}$ of lowest possible degree such that there exists $\pi : \tilde{S} \to S$ with

$$\begin{tikzcd} \tilde{S} \arrow[d, "\pi"'] \arrow[dr, "\tilde{f}"] & \\ S \arrow[r, "f"'] & \mathbb{P}^1\mathbb{C} \end{tikzcd}$$ commuting.

([Girondo, Gonzalez-Diez] contains some explicit constructions of normalisations).

Cor 2.73: $\mathrm{Mon}(f) \cong \mathrm{Aut}(\tilde{S}, \tilde{f})$

Interpretation in terms of Fuchsian groups

Prop 2.74: Let $f : S_1 \to S$ be a covering of RS's, with $S_1 \setminus f^{-1}(\{y_1, \dots, y_n\}) \longrightarrow S \setminus \{y_1, \dots, y_n\}$ the corresponding unramified covering. Let $\pi : \mathbb{H}/\Gamma_1 \longrightarrow \mathbb{H}/\Gamma$ be the Fuchsian group representation. The normalisation of f can be represented as the compactification of

$$\mathbb{H} \Big/ \bigcap_{\gamma \in \Gamma} \gamma^{-1}\Gamma_1\gamma.$$ Covering group $G \cong \Gamma \Big/ \bigcap_{\gamma \in \Gamma} \gamma^{-1}\Gamma_1\gamma \cong \mathrm{Mon}(f)$.

<u>Example:</u> Let $F(X,Y) = Y^2 X - (Y-1)^3$, $S_F = \{(X,Y) \mid F(X,Y) = 0\}$

Consider $\underline{x} : S_F \longrightarrow \mathbb{P}^1\mathbb{C}$

$(x,y) \longmapsto x$

In fact, ① S_F has genus 0

② $\underline{x} : S_F \longrightarrow \mathbb{P}^1\mathbb{C}$ is of degree 3 and ramified at most over $\{0, -27/4, \infty\}$.

③ $\mathrm{Mon}(\underline{x}) \simeq \Sigma_3 \Rightarrow$ not normal covering

④ Normalisation of $(S_F, \underline{x})$ is $(S_{\tilde{F}}, \tilde{\underline{x}})$ where

$$\tilde{F}(X,Y) = Y^2(1-Y)^2 X + (1-Y+Y^2)^3$$
$$= F(X, Y(1-Y)).$$