BUNTES

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Dessins d'Enfants

Goal: Define an action $G_{\mathbb{Q}} \curvearrowright (X, D)$ $\longleftrightarrow$ An action $G_{\mathbb{Q}} \curvearrowright (S, \beta)$

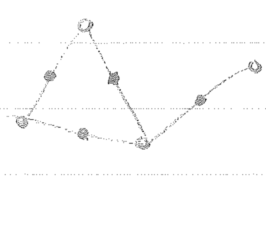
(X,D): Dessin; via valuations; (S,β): Belyi pair

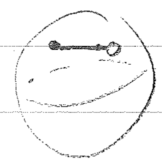
① Define Dessins
② $\longrightarrow$
③ $\longleftarrow$
④ Action on valuations

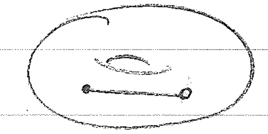
§4.1. Dessins

Dfn: A dessin is a pair (X, D) where X is an oriented compact topological surface and $D \subseteq X$ is a finite graph such that
① D is connected
② D is bicoloured
③ $X \setminus D = \coprod$ (finitely many topological disks), call them the faces of D

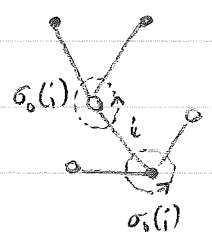
Remarks: • ③ $\Rightarrow$ ①
• Any graph can be made into a Dessin by putting extra vertices in the middle of every edge and colouring them differently



Example : 

Non-example : 

Permutation representation of a Dessin

D 

Label the edges by $\{1, \dots, N\}$

$\sigma_0(i)$ = draw a small circle around the white vertex on the edge i, then $\sigma_0(i)$ is the first vertex you hit travelling around the circle with +ve orientation

$\sigma_1(i)$ = same, but around the black vertex

Dfn: (σ_0, σ_1) is the permutation representation pair of (X, D).

Say $\sigma_0 = (1 \dots N_1)(N_1+1 \cdots N_2)(\dots) \dots$, a product of disjoint cycles

Each cycle $(N_j+1 \cdots N_{j+1})$ corresponds to a white vertex.
length of the cycle = degree of the corresponding vertex.

Same for σ_1 and black vertices.

Fact: $\left\{ \begin{array}{c} \text{Cycles appearing in the} \\ \text{decomposition of } \sigma_0\sigma_1 \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Faces of} \\ D \end{array} \right\}$

Pf: Exercise.

Remark:
- D connected $\Rightarrow$ $\langle \sigma_0, \sigma_1 \rangle$ is transitive in Σ_N $\left(N = \text{number of vertices} \right)$
- D bicolored $\Rightarrow$ cycles on D contain an even number of edges

Fact: In general, a dessin is not a triangulation of X (consider the example above), but nonetheless

$$\chi(X) = v - \underbrace{e}_{=N} + f$$

Pf: Later.

Prop: $\chi(X) = (\#\text{ cycles } \sigma_0 + \#\text{ cycles } \sigma_1) - N + (\#\text{ cycles of } \sigma_0\sigma_1)$

Recovering the Dessin from the permutation representation $\left((\sigma_0, \sigma_1) \rightsquigarrow (X, D) \right)$

(σ_0, σ_1) s.t. $\langle \sigma_0, \sigma_1 \rangle \leq \Sigma_N$ is transitive

Prop: There exists (X, D) with permutation representation (σ_0, σ_1)

Pf: Write $\sigma_0\sigma_1 = \tau_1 \cdots \tau_k$, τ_i disjoint cycles, each of length n_i (with $\sum n_i = N$)

Create faces bounded by $2n_1, \ldots, 2n_k$ vertices.

Assign the vertices white and black colours so that the graph is bicoloured.

$2n_1$ $\qquad\qquad$ $2n_k$

Select an edge i and label the others as prescribed by $\sigma_0 \sigma_1$.
Repeat for each face.
Glue edges with the same σ-labels.
$\Rightarrow$ Get (X, D). $\square$

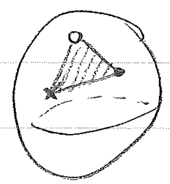
<u>Dfn:</u> $(X_1, D_1) \sim (X_2, D_2)$ if there exists an orientation preserving homeomorphism $\varphi : X_1 \to X_2$ s.t. $\varphi|_{D_1} : D_1 \xrightarrow{\sim} D_2$.

<u>Thm:</u> $\{\text{Dessins}\}/\sim \;\longleftrightarrow\; \{(\sigma_0, \sigma_1) \mid \langle \sigma_0, \sigma_1 \rangle \leq \Sigma_N \text{ transitive}\}$

<u>§4.2. Dessins $\longrightarrow$ Belyi pairs</u>

One can compute a <u>triangle decomposition</u> of (X, D).
This gives rise to $T(D)$, where we remember orientations.

<u>Ex:</u>



2 triangles with 3 common edges.

Given $T(D)$, we construct a Belyi map

$$f_D : X \longrightarrow \hat{\mathbb{C}}.$$

To see this, for each edge j we have a pair of triangles $T_j^{\pm}$ adjacent to it, with opposite orientation.

Then get $f_j^{\pm} : T_j^{\pm} \longrightarrow \overline{\mathbb{H}}^{\pm}$ such that $f_j^+ = f_j^-$ on $T_j^+ \cap T_j^-$ (the edge).

and

$$f_j^{\pm}\big|_{\partial T_j^{\pm}} : \partial T_j^{\pm} \longrightarrow \mathbb{R} \cup \{\infty\}$$

$$\circ \longmapsto 0$$

$$\bullet \longmapsto 1$$

$$\times \longmapsto \infty$$

$\mathrm{Branch}(f_D) = \{0, 1, \infty\}$

$\deg(f_D) = \#$ edges of D

$m_v(f_D) = \deg v$

$f_D^{-1}([0,1]) = D$

We can in fact find a Riemann surface S_D such that S_D is homeomorphic to X.

Then $f_D : S_D \longrightarrow \mathbb{P}^1$ and we have a Belyi map.

<u>Dfn</u>: Let (S, f) be a Belyi pair.

We say $(S_1, f_1) \sim (S_2, f_2)$ if they are isomorphic as ramified coverings.

So we have $\{(S, f)\} \longleftrightarrow \{(X, D)\}$.

$\{\text{Dessins}\}/\sim \;\longleftrightarrow\; \{\text{Belyi pairs}\}/\sim$

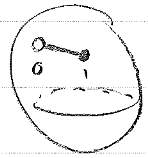
Using this we define the Galois action on Dessins by

$$\begin{tikzcd}
(X, D) \arrow[r, dashed] \arrow[d, squiggly] & (X, D)^\sigma \\
(S_D, f_D) \arrow[r, squiggly] & (S_D^\sigma, f_D^\sigma) \arrow[u, squiggly]
\end{tikzcd}$$

Some Galois invariants:
- # edges, vertices, faces
- genus
- automorphism group

This action of Galois on Dessins is faithful.

Example: An example of $(X, D) \rightsquigarrow (S_D, f_D)$

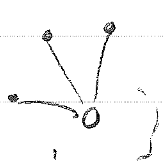


$f_D = z$

$\deg f_D = \#$ edges

$m_v(f_D) = \deg v$

Exercise: ①  find f_D

②  n copies