BUNTES

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A Sandwich Table of Dessins d'Enfants

Recall: We have bijections

$$\left\{\begin{matrix}\text{Belyi pairs } (S,f) \\ \deg f = d\end{matrix}\right\}/\sim \;\longleftrightarrow\; \left\{\begin{matrix}\text{Dessins with} \\ d \text{ edges}\end{matrix}\right\}/\sim$$

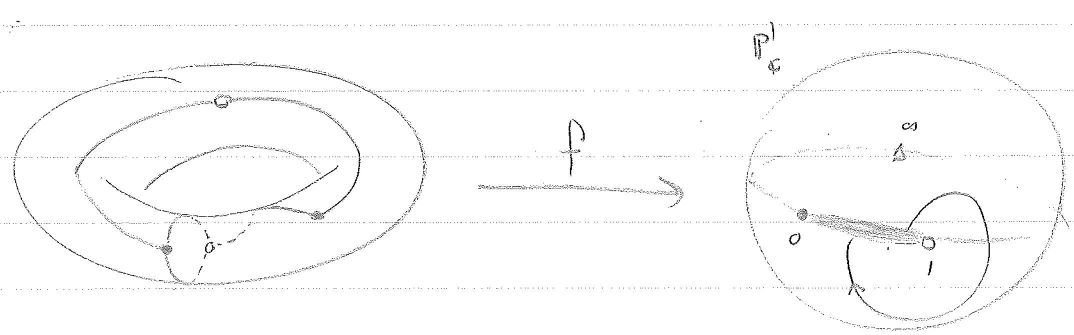
$$\longleftrightarrow \left\{\begin{matrix}\deg d \text{ (unramified)} \\ \text{covers of } \mathbb{P}^1\setminus\{0,1,\infty\}\end{matrix}\right\}/\sim$$

$$\longleftrightarrow \left\{\begin{matrix}\text{Conjugacy classes of subgroups of} \\ \pi_1(\mathbb{P}^1_{\mathbb{C}}\setminus\{0,1,\infty\}, \overline{01}) \text{ of index } d\end{matrix}\right\}$$

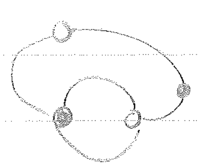
$\overline{01} = (0,1)$ open interval

$$\longleftrightarrow \left\{\begin{matrix}\text{Conjugacy class of homomorphisms from } F_2 \text{ onto} \\ \text{a transitive subgroup of } S_d\end{matrix}\right\}$$

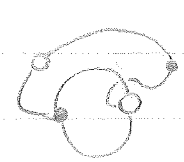
$$\longleftrightarrow \left\{\begin{matrix}\text{Conjugacy classes of ordered pairs of} \\ \text{permutations generating a transitive subgroup of } S_d\end{matrix}\right\}$$



Note: This is not the same as the Dessin on the left below.



Ordering around a vertex matters. Thus the right ⇝



We have

$$1 \longrightarrow \pi_1^{\acute{e}t}\left(\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0,1,\infty\}\right) \longrightarrow \pi_1^{\acute{e}t}\left(\mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,\infty\}\right) \longrightarrow \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow 1$$

and thus get an action $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \curvearrowright \pi_1^{\acute{e}t}\left(\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0,1,\infty\}\right)$.

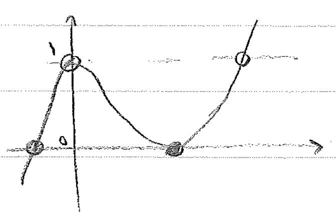
<u>Goal</u>: Construct invariants of the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on Dessins.

One such invariant is the number of black/white vertices / faces.
Valency sequences.

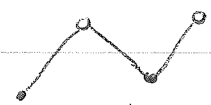
<u>Simplest Dessins - Trees in $\mathbb{P}^1$</u>

These correspond to Belyi maps given by polynomials with 2 branch values.
Shabat polynomials / Generalised Chebyshev polynomials

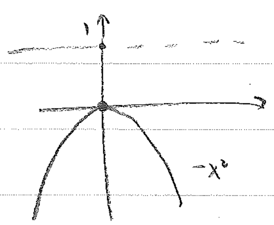
<u>Examples</u>: (1)



$\longrightarrow$ Dessin



(2)

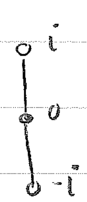


This is over $\mathbb{R}$, so the picture makes it seem like 1 has no preimage

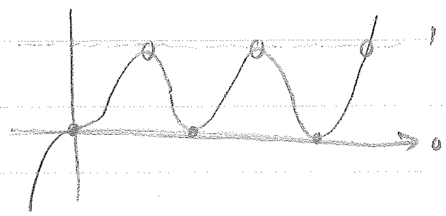
$-x^2 = 1$

$x^2 = -1, \quad x = \pm i$

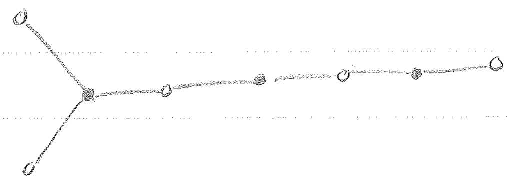
So the Dessin is



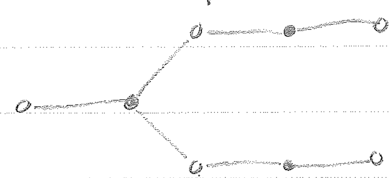
(3) Can we combine these?



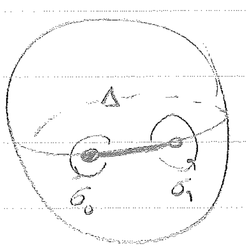
deg 7

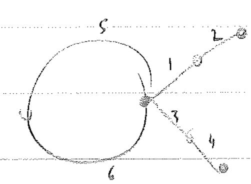


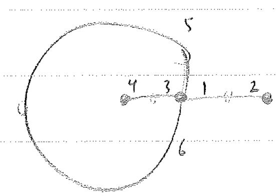
We can write down a Belyi function for this over $\mathbb{Q}(\sqrt{21})$.
Thus Galois should act, preserving the invariants above, but giving a nonisomorphic Dessin.



Beyond Trees







$\sigma_0 = (1563)$

$\sigma_1 = (12)(34)(56)$

$\# M = 120$

$S_5 \simeq PGL_2(\mathbb{F}_5) \subseteq S_6$

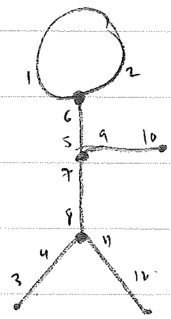
$\sigma_0 = (1536)$

$\sigma_1 = (12)(34)(56)$

$\# M = 24$

$\overset{\shortparallel}{S_4} \subseteq S_6$

(embedded in a transitive way)



$\sigma_1 = (1\,2)(3\,4)(5\,6)(7\,8)(9\,10)(11\,12)$

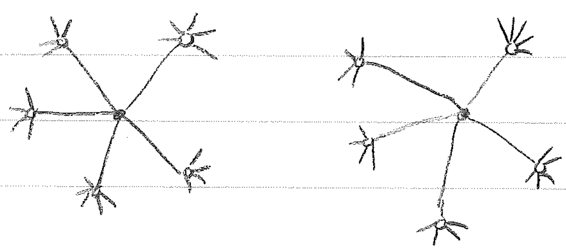
$\sigma_0 = (1\,6\,2)(4\,11\,8)(5\,7\,9)$

Which subgroup of S_{12} is this?

The Mathieu group M_{12}

(Fact: Every nonabelian finite simple group is generated by 2 elements, one of which is order 2.
Thus we can play this game.)

The Belyi map corresponding to the above Dessin is defined over $\mathbb{Q}(\sqrt{-11})$.
In fact this is also the field of definition of the character table of M_{12}.

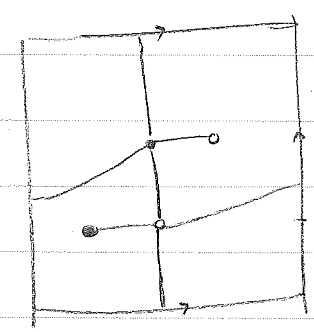


These both have monodromy group $M = A_{27}$
They are defined $/\mathbb{Q}$.
However, they are nonisomorphic.

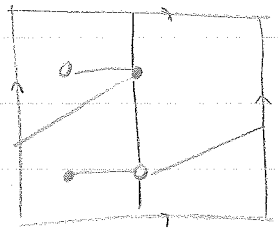
Some examples in genus 1

A deg 5 example: Cycle types – $\sigma_0 = 4^1 1^1$

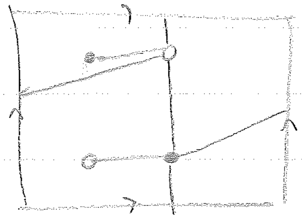
$\sigma_1 = 4^1 1^1$



Monodromy S_5



$\# M = 20$



$\# M = 20$

Let's find a Belyi map

$$E: y^2 = x^3 + ax + b \xrightarrow{\varphi} \mathbb{P}^1$$
$$\infty \longmapsto \infty$$
$$(x,y) \longmapsto cxy + dy + fx^2 + gx + h$$

$$\varphi(x,y) = 0 \Rightarrow y = \frac{-fx^2 - gx - h}{cx + d}$$

Need quartic pt & single pt above 1, 0.

$$(*)^2 - (x^3 + ax + b) = 0$$

should factor as

$$(-fx^2 - gx - h)^2 - (cx+d)^2(x^3 + ax + b) = c^2(x-p)^4(x-q).$$

Similarly for 1, $\exists r, s$ similarly.

To solve, fix $c = 1$.

$$a = \frac{5}{16},\ b = \frac{5}{32},\ c = 1,\ d = \frac{-5}{4},\ f = g = 0,\ h = \frac{1}{2}$$

Other solutions are defined $/ \mathbb{Q}(i)$.