# RS and discrete groups

Facts:

(Prop 1.27)
- $\mathrm{Aut}(\hat{\mathbb{C}}) = \{ z \mapsto \frac{az+b}{cz+d} \} = PSL(2,\mathbb{C})$
- $\mathrm{Aut}(\mathbb{C}) = \{ z \mapsto az+b \}$
- $\mathrm{Aut}(\mathbb{H}) = \{ z \mapsto \frac{az+b}{cz+d},\ abcd \in \mathbb{R} \} = PSL(2,\mathbb{R})$

(Thm 1.69)
- $\Sigma$ has a universal cover $\tilde{\Sigma}$, $\pi_1(\tilde{\Sigma}) = 0$
- $\Sigma = \tilde{\Sigma}/G$, $G \subset \mathrm{Aut}(\tilde{\Sigma})$
- $G \cong \pi_1(\Sigma)$ acts freely and properly discontinuously on $\tilde{\Sigma}$

{Goal: learn about $\Sigma$ by looking at $\tilde{\Sigma}/G$} later

## 2.1 Uniformization

Thm (Unif.)

The only simply connected RS are (iso.) $\mathbb{C}, \hat{\mathbb{C}}, \mathbb{H}$.

Thm

Let $\Sigma$ be a compact RS. If
- $g=0$ : $\Sigma \cong \hat{\mathbb{C}}$
- $g=1$ : $\Sigma \cong \mathbb{C}/\Lambda$
- $g=2$ : $\Sigma \cong \mathbb{H}/K$

Pf
- $g=0$ (Unif)
- $g \geq 1$, by RH $\hat{\mathbb{C}}$ can't be universal cover. $(2g-2) = d(2g'-2) + \sum_{x=X} (m_x(f)-1)$
  - $g=1$ : no $\underset{\pi_1(\Sigma)}{\mathbb{Z} \oplus \mathbb{Z}} \cong H < PSL(2,\mathbb{R})$ acting free & p.d.
  - $g=2$ : free subgroups of $\mathbb{C}$ are abelian $\{z \mapsto az+b\}$
    $\pi_1(\Sigma)$ not abelian for $g \geq 2$

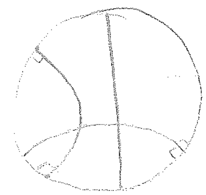
# Fuchsian groups

If $g \geq 0$, $\Sigma = \mathbb{H}/\Gamma$ with $\Gamma < PSL(2,\mathbb{R})$ $\uparrow$ $\mathrm{Isom}^+\left(\mathbb{H}, \frac{|dz|^2}{\mathrm{Im}\,z}\right)$

Digress about

- $\mathrm{Aut}(\mathbb{H}) = PSL(2,\mathbb{R})$ $\left\{\begin{array}{l}\cdot \text{ also the group of hyperbolic isometries} \\ \cdot \text{ transitive on geodesics.}\end{array}\right.$
  - geosics are

or



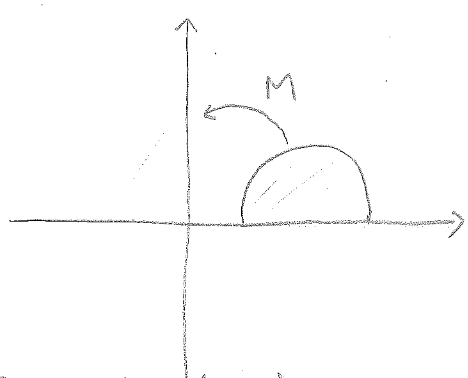
- Quotient mfd thm requires $\Gamma$ to act free and properly discontinuously

Def: A Fuchsian grp $\Gamma$ = discrete subgrp of $PSL(2,\mathbb{R})$. Fact: $\mathbb{H} \to \mathbb{H}/\Gamma$ is holomorphic cover

## Triangle groups

## Reflections on $\mathbb{H}$

$PSL(2,\mathbb{R})$ acts transitively on geodesics on $\mathbb{H}$, let $L$ be a geo, and $M \in PSL(2,\mathbb{R})$ st. $M(L) = \mathrm{Im} = \{x=0\}$




then $R = M^{-1} \circ (-\bar{z}) \circ M$ is a reflection along $L$

$(-\bar{z})$ is a reflection along Im.

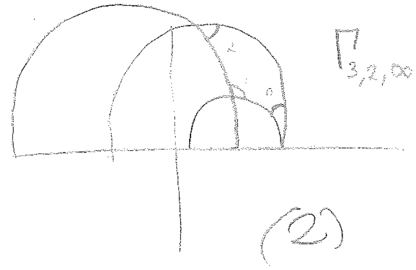
these are anticonformal isometries, hence $R \circ (-\bar{z}) \in PSL(2,\mathbb{R})$

so $\boxed{R(z) = \frac{a\bar{z}+b}{c\bar{z}+d}}$

Construct Fuchsian gps by reflecting polygons (Tesselation)

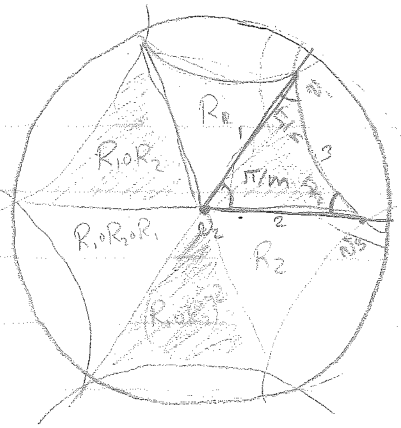
## $\Delta$ groups

Given $m, n, \ell \in \mathbb{Z} \cup \{\infty\}$ consider the $\Delta$ with angles $\frac{\pi}{m}, \frac{\pi}{n}, \frac{\pi}{\ell}$



$\Gamma_{3,2,\infty}$

$\alpha = \frac{\pi}{3}$, $\beta = \pi/2$, $\gamma = 0$




$R_1$ = reflection along $\overline{v_1 v_2}$

$R_2$ = reflection along $\overline{v_2 v_3}$

$R_3$ = " " $\overline{v_3 v_1}$

so $R_i$ tesselate $\mathbb{D}$ but $R_i \notin PSL(2,\mathbb{R})$

sol: $x_1 = R_3 \circ R_1$, $x_2 = R_1 \circ R_2$, $x_3 = R_2 \circ R_3$

now $x_i \in PSL(2,\mathbb{R})$ and satisfy $x_1^n = x_2^m = x_3^l = x_1 x_2 x_3 = 1$

Fundamental domain is ▱ quadrilateral

Def.

let $\Gamma_{n,m,l} = \cancel{\langle x_1, x_2 \rangle = \langle x_1, x_3 \rangle = \langle x_2, x_3 \rangle}$

$= \langle x_1, x_2, x_3 \,;\, x_1^n = x_2^m = x_3^l = x_1 x_2 x_3 = 1 \rangle$

if $n, m, l = \infty \Rightarrow$ no relation. Triangle group of signature $(n,m,l)$

Remark

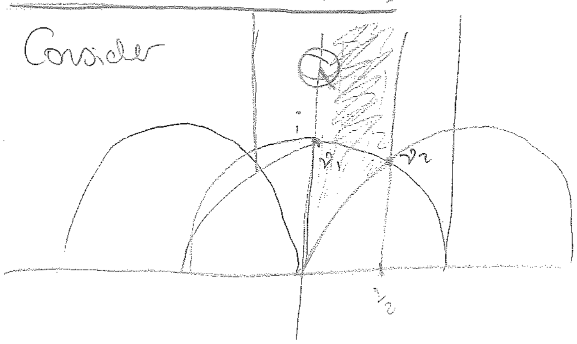
for $\frac{1}{n} + \frac{1}{m} + \frac{1}{l} = 1$ tesselate $\mathbb{R}^2$

$\frac{1}{n} + \frac{1}{m} + \frac{1}{l} > 1$ tesselate $S^2$

Example I

Modular gp $PSL(2,\mathbb{Z})$

Consider




has angles $\frac{\pi}{2}, \frac{\pi}{3}, 0$ so $n, m, l = 2, 3, \infty$

$R_1(z) = \frac{1}{\bar{z}}$, $R_2(z) = -\bar{z} + 1$

$R_3(z) = -\bar{z}$

so

$x_1(z) = -1/z$, $x_2(z) = \frac{1}{-z+1}$

$x_3(z) = z+1$

Thm

$\Gamma_{2,3,\infty} \cong PSL(2,\mathbb{Z}) = \frac{SL(2,\mathbb{Z})}{\langle \pm Id \rangle}$

Example II $\Gamma(2)$

Obs (Lemma 2.32)

$\Gamma_1 < \Gamma_2$, $\gamma_1, \dots, \gamma_n \in \Gamma_2$ representatives of $\Gamma_2 \backslash \Gamma_1$

then if $Q$ is a FD for $\Gamma_2$, $Q' = \bigcup_{i=1}^n \gamma_i(Q)$ is a FD $\Gamma_1$

~~Example 2.33~~

$\Gamma(2) \subset PSL(2,\mathbb{Z})$ congruence subgroup of level 2.

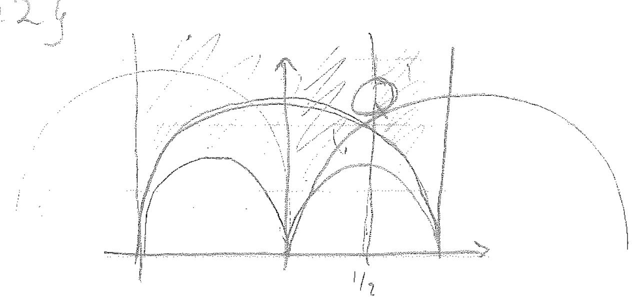
$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \mathrm{Id} \bmod 2 \right\}$$

claim corresponds to $\Gamma_{\infty,\infty,\infty}$

$|\Gamma(1) : \Gamma(2)| = 6$

$\mathrm{Id} = z$, $\gamma_2 = \frac{-1}{z-1}$, $\gamma_3 = \frac{z-1}{z}$

$\gamma_4 = \frac{z-1}{1}$, $\frac{-z}{z-1}$, $\frac{-1}{z}$




eg.

$$z = e^{i\theta} \quad \gamma_2(z) = \frac{-1}{e^{i\theta}-1} = \frac{-e^{-i\theta}+1}{2-2\cos\theta} \qquad Re(\gamma_2(z)) = \frac{1-\cos\theta}{2-2\cos\theta} = \frac{1}{2}$$

So corresponds to $\Gamma_{\infty,\infty,\infty}$ $\langle x_1, x_2, x_3 \mid x_1x_2x_3 = 1 \rangle = \pi_1(S^1 \setminus 3\text{ pts})$

~~Example 2.34~~

$\rightsquigarrow$ can generalise this to $\Gamma(n)$

z

## Automorphisms of RS

- **Prop.**

  $S_1 = \mathbb{H}/\Gamma_1 \cong S_2 = \mathbb{H}/\Gamma_2 \iff \exists_1 T \circ \Gamma_1 \circ T^{-1} = \Gamma_2$, $T \in PSL(2,\mathbb{R})$

  ($\Gamma_i$ uniformizing $S_i$)

**Pf**

$\Leftarrow$ Isomorphism given by $[z]_1 \longmapsto [T(z)]_2$

$\Rightarrow$ lift $\phi: S_1 \to S_2$ to $\tilde{\phi}: \mathbb{H} \to \mathbb{H}$ and put $T = \tilde{\phi}$

- **Prop.**

  $\Gamma$ Fuchsian & free on $\mathbb{H}$,

  $$\mathrm{Aut}(\mathbb{H}/\Gamma) = N(\Gamma)/\Gamma$$

**Pf**

From previous prop $T \in N(\Gamma)$ ($T \circ \Gamma \circ T^{-1} = \Gamma$) gives an $\mathrm{Aut}(\mathbb{H}/\Gamma)$

$$N(\Gamma) \to \mathrm{Aut}(\mathbb{H}/\Gamma)$$

Kernel is $\Gamma$.

- **Cor.**

  Automorphism gp of a cp RS of $g \geq 2$ is finite

**Pf**

$S = \mathbb{H}/\Gamma$. by

$$\begin{array}{ccc} & \mathbb{H} & \\ & \downarrow & \searrow \\ S = \mathbb{H}/\Gamma & \xrightarrow{f} & \mathbb{H}/N(\Gamma) = S/\mathrm{Aut}(S) \\ [z]_\Gamma & \longmapsto & [z]_{N(\Gamma)} \end{array}$$

$\deg f = |N(\Gamma)/\Gamma|$ and $\deg f$ holomorphic $< \infty$

$$f^{-1}([z]_{N(\Gamma)}) = \{([\gamma_n(z)]_\Gamma)\}$$

Say now given $\Sigma$ of $g \geq 2$ and $G < \mathrm{Aut}(S)$ and let $\tilde{g}$ be the genus of $\Sigma/G$, then by RH

$$2g-2 = |G|(2\tilde{g}-2) + \sum_{i=1}^{n}(|I(p)|-1) = |G|$$

where $I(p)$ is the stabilizer of $p$ in $G$.

**Exercise** (maybe not that hard now)

$$\mathrm{Aut}(\Sigma) \leq 84(g-1)$$

Hint (genus).

**Exercise**

$1 \to \Gamma(n) \to \Gamma(1) \to PSL[illegible] \to 1$

compute genus of $S_n$