1 Riemann Surfaces - Sachi Hashimoto

1.1 Riemann Hurwitz Formula

We need the following fact about the genus:

Exercise 1 (Unimportant)**.** The genus is an invariant which is independent of the triangulation. Thus we can speak of it as an invariant of the surface, or of the Euler characteristic $\chi(X) = 2 - 2g$.

This can be proven by showing that the genus is the dimension of holomorphic one forms on a Riemann surface.

Motivation: Suppose we have some hyperelliptic curve $C : y^2 = (x+1)(x-1)(x+2)(x-2)$ and we want to determine the topology of the solution set. For almost every $x_0 \in \mathbb{C}$ we can find two values of $y \in \mathbb{C}$ such that (x_0, y) is a solution. However, when $x = \pm 1, \pm 2$ there is only one y-value, $y = 0$, which satisfies this equation. There is a nice way to do this with branch cuts–the square root function is well defined everywhere as a two valued functioned except at these points where we have a portal between the "positive" and the "negative" world. Here it is best to draw some pictures, so I will omit this part from the typed notes.

But this is not very systematic, so let me say a few words about our eventual goal. What we seem to be doing is projecting our curve to the x-coordinate and then considering what this generically degree 2 map does on special values. The hope is that we can extract from this some topological data: because the sphere is a known quantity, with genus 0, and the hyperelliptic curve is our unknown, quantity, our goal is to leverage the knowns and unknowns. The idea is that there is an exact relation between the genus of the surfaces we are mapping between, the degree of the map, and the number of branch points, so that changing any one would force a change in the others. Here there are 4 branch points, it is degree 2, the sphere has Euler characteristic 2 and the hyperelliptic curve has Euler characteristic 0. This makes it tempting to put the numbers together into the formula $0 = 2 \cdot 2 - 4$, which would tell us that $\chi(C) = \deg(\pi)\chi(\mathbb{P}^1\mathbb{C}) - b$ where b is the branching and π is the projection map.

This formula, or its generalization, is called the Riemann-Hurwitz formula is an incredibly useful formula that relates the genus of two Riemann surfaces to the degree of a map $f : S_1 \to S_2$ and the branching of f.

Let $P(x, y)$ be a nonsingular irreducible polynomial in x and y and write $P(x, y) = y^n + p_{n-1}(x)y^{n-1} + \cdots + p_0(x)$ by making an affine linear change of coordinates over $\mathbb{C}$. Consider the Riemann surface cut out by P, $R = \{(x, y) | P(x, y) = 0\}$ and the projection $\pi : R \to \mathbb{P}^1\mathbb{C}$ given by $\pi(x, y) = x$ to the x-coordinate. Above most x-values $x_0 \in \mathbb{P}^1\mathbb{C}$ there are n distinct solutions to $P(x_0, y)$ however there are certain values which have fewer than n solutions. These are the *branch points.* Note that there are only finitely many such points, and they are given by the roots of the discriminant of P.

We will define a quantity, the *total branching index* b of P to be

$$\sum_{x \in \mathbb{P}^1\mathbb{C}} (\deg(\pi) - |\pi^{-1}(x)|)$$

which we noted was finite.

Then we claim that

$$\chi(R) = \deg(\pi)\chi(\mathbb{P}^1\mathbb{C}) - b.$$

To prove this we will use the dual nature of the Euler characteristic: both that Euler characteristic is invariant of the triangulation put on the surface, and that we can compute it given a triangulation by the formula vertices minus edges plus faces.

First we need to understand a property about maps of complex surfaces.

Lemma 1. *Locally, given some choice of coordinates, a nonconstant morphism of Riemann surfaces $f : R \to S$ can be written $w \mapsto w^m$. More precisely:*

Let $r \in R$ and $f(r) = s$. Take V_s to be a sufficiently small neighborhood of s and choose an identification of V_s with the unit disc D sending $s \mapsto 0$, $\psi : V_s \to D$. Then we can also find an analytic identification $\phi : U_r \to D$ of a neighborhood of r with the disc such that $f(U_r) \subseteq V_s$ and the following diagram commutes:

$$\begin{array}{ccc} U_r & \xrightarrow{\quad f \quad} & V_s \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \psi} \\ D & \xrightarrow[w \mapsto w^m]{} & D \end{array}$$

That is, $(\psi \circ f)(x) = \phi(x)^m$ for all $x \in U_r$.

Proof. Choose a local chart centered at r, $h : U_r \to D$. Then $g := \psi \circ f \circ h^{-1}$ is analytic near 0. Let m be the order of the zero, and write $g(z) = g_m z^m + O(z^{m+1})$. Then we can define $g^{1/m}$ near zero, this is of the form $g_n^{1/m} z + O(z^2)$. Since the derivative at 0 does not vanish, the mth root of g is a local analytic isomorphism near zero , but $g^{1/m} \circ h$ is a local analytic isomorphism from a neighborhood of r to a neighborhood of 0 and its nth power is $\psi \circ f$. Shrinking V_s if necessary and rescaling ψ appropriately take $\phi = g^{1/m}$.

Note that the number m does not depend on the choice of neighborhoods as it is simply the solutiosn to $f(x) = y$ as y goes to s which are contained in a small neighborhood of r. □

As Ricky stated last time, every Riemann surface admits a triangulation. We can compute the Euler characteristic of a Riemann surface by $\chi(R) = V - E + F$ where V, E and F count vertices, edges, and faces of a triangulation of R. The Euler characteristic is also equal to $\chi(R) = 2 - 2g$ where g is the genus.

From these facts we will derive our proof of Riemann Hurwitz for projections. First, construct a triangulation of $\mathbb{P}^1\mathbb{C}$ (the image) which pulls back to a triangulation of R, by the lemma, which we will require to include all of the branch points as vertices of the triangulation. Each face lifts to $\deg(\pi)$ other faces and each edge lifts to $\deg(\pi)$ other edges. Each vertex also lifts to $\deg(\pi)$ other vertex except for the branch points x_0, which lift to $|\pi^{-1}(x_0)|$ vertices by definition.

Thus $\chi(R) = \deg(\pi)\chi(\mathbb{P}^1\mathbb{C}) - \sum_{x \in S}(\deg(\pi) - |\pi^{-1}(x)|)$.

Example 1. This gives us a way to compute the genus of a smooth irreducible curve. Let $P(x, y)$ be a degree d curve. Then $\chi(R) = 2g - 2$ and $\chi(\mathbb{P}^1\mathbb{C}) = 2$ so it remains to calculate the number of branch points and their multiplicity.

In order to do this, we use some classical algebraic geometry. The projective space $\mathbb{P}^2 = [x : y : z]$ has a dual projective space $[a : b : c]$ which we can think of as the space of lines in $\mathbb{P}^2$. One way to consider this is the following: given a (homogeneous) equation for a line $ax + by + cz = 0$ there is no distinction between variables x, y, and z and coefficients a, b, and c, so we can switch between thinking about the space of variables or the space of coefficients. The normal projective space $\mathbb{P}^2$ is the space of points $[x : y : z]$ and the dual projective space $(\mathbb{P}^2)^*$ is the space of lines $[a : b : c]$. Given a point, we can associate to it the set of lines through that point, and given a line, we can associate to it the set of points on that line, this gives us maps from $\mathbb{P}^2$ to $(\mathbb{P}^2)^*$ and back, taking a point to a line and a line to a point. Furthermore, given a smooth curve C in $\mathbb{P}^2$ we can associate a dual curve C^* by taking each point Q to the tangent line t_Q at that point.

To compute the branching index of a polynomial $P(x, y)$, it would be enough to show that the dual curve to $P(x, y)$ has degree $d(d-1)$. In other, words if we fix an arbitrary point on our curve then there are $d(d-1)$ lines through it which are tangent lines to the curve. Geometrically, we can think of projecting in some direction as sweeping a parallel family of lines across a curve and considering how many points we intersect: this map is branched if the line is tangent to any point. Note that a family of parallel lines in projective space is the same as lines projecting from a point at infinity. So, in the dual projective space, sweeping a family of lines corresponds to taking a single line, intersecting with the dual cruve, and considering how many points lie on that line, counting with multiplicity. These points in the dual $\mathbb{P}^2$ correspond to the tangencies in $\mathbb{P}^2$, and the line in the dual space corresponds to the family of parallel lines given by the map from points to lines described above. Thus by Bezout, in order to compute the branching of the projection, it would be equivalent to compute the degree of C^*.

The following proof is due to Matt Emerton. Consider an arbitrary point on our curve in $\mathbb{P}^2$ such that none of the tangent lines to the points at infinity pass through the point and change coordinates so that this point lies at the origin. Rewrite our equation as $P(x, y) = f_d + f_{d-1} + \cdots + f_0$ where the f_i are the ith homogneous part. Then we know that f_{d-1} is coprime to f_d because of the condition about tangent lines at infinity. In particular, if they were not coprime then the asymptotes are exactly given by the vanishing of linear factors of the highest degree term, $f'_{d,x}x + f'_{d,y}y + f_{d-1} = 0$ where these denote the partials with respect to x and y of f_d. Then whichever linear factor they share will divide this, and thus this gives a line through the origin which is tangent at infinity.

Then $(0, 0)$ is on the tangent line through (x_0, y_0) if and only if $x_0P_x(x_0, y_0)+y_0P_y(x_0, y_0)$ vanishes where P_x and P_y are the partial derivatives of P with respect to x and y, so we are trying to count the solutions to the simultaneous system of equations $P = 0$ and $xP_x + yP_y = 0$.

In terms of the homogeneous components we can write

$$f_d + f_{d-1} + \cdots + f_0 = 0$$

and

$$df_d + (d-1)f_{d-1} + \cdots + f_1 = 0$$

and then rewrite as

$$f_d + f_{d-1} + \cdots + f_0 = 0$$

and

$$f_{d-1} + 2f_{d-2} + \cdots + (d-1)f_1 = 0.$$

By Bezout's theorem, simultaneous equations of degree d and $d-1$ have $d(d-1)$ common solutions, and they are all affine since f_d and f_{d-1} are coprime. So the dual curve has degree $d(d-1)$, that is, there are $d(d-1)$ lines tangent to C passing through the origin.

Applying Riemann Hurwitz, with $b = d(d-1)$ we can compute that

$$g = \frac{(d-1)(d-2)}{2}.$$

More generally, this strategy will work on moprhisms from $f : R \to S$ where we replace S with any Riemann surface.

Theorem 1 (Riemann-Hurwitz). *Given $f : R \to S$ a morphism of compact Riemann surfaces. Then*

$$\chi(R) = \deg(f)\chi(S) - \sum_{x \in S} (\deg(f) - |f^{-1}(x)|).$$

The same strategy works, with the obvious adaptations.

Example 2. There are no nonconstant morphisms from a sphere to a surface of genus greater than zero. If X and Y are Riemann surfaces and $\chi(X) = 2$ and $\chi(Y) \leq 0$, but $f : X \to Y$ is a nonconstant holomorphic map then we have that $2 = \deg(f)\chi(Y) - b$. This formula will never hold, as the degree and the branching index are always positive while $\chi(Y) \leq 0$.

Example 3. The Fermat equation $x^n + y^n = z^n$ is not solvable in nonconstant polynomials. Write $x^n + y^n = 1$, then our goal is to show that this is not solvable in rational functions of t. Let $P(x, y) = x^n + y^n - 1$ and let R be the Riemann surface cut out by P. Note that R is nonsingular. Let $\pi : R \to \mathbb{P}^1\mathbb{C}$ be the projection $(x, y) \mapsto x$. We will use this map to determine the genus of R.

Then if ζ is a primitive n-th root of unity, $P(x, y)$ has branch points precisely at the values $x = \zeta, \zeta^2, \ldots, \zeta^n$ because the only y value above these x-values is 0.

Thus there are n branch points each of which has one point in the preimage under π so $b = (n-1)n$. Applying Riemann-Hurwitz, we get

$$\chi(R) = n\chi(\mathbb{P}^1\mathbb{C}) - (n-1)n$$

or simply

$$\chi(R) = 2n - (n^2 - n) = 3n - n^2.$$

In order to solve the Fermat equation in polynomials we would need a surjective rational map from $\mathbb{P}^1\mathbb{C}$ to R. By the previous example, we know that such a map can only exist when R has genus 0. However, using this formula, that can happen if and only if $n = 1$ or $n = 2$.

2 A Threefold Equivalence of Categories

Taking a page out of Mumford's *Curves and Their Jacobians*: "The beginning of the subject is the AMAZING SYNTHESIS, which surely overwhelmed each of us as graduate students and should really not be taken for granted. Starting in 3 distinct fields of mathematics, we can consider three types of objects..."

1. Analysis: Compact connected Riemann surfaces
2. Algebra: Field extensions of $K/\mathbb{C}$ where K is finitely generated and transcendence degree one over $\mathbb{C}$
3. Geometry: Complete, non-singular, irreducible algebraic curves inside of $\mathbb{P}^n$

To get from a curve to a field extension we can associate to a curve C the field K of functions $f : C \to \hat{\mathbb{C}}$ given by rational functions p/q on C with $\deg p = \deg q$. The Riemann surface associated to C is just the surface with the induced complex structure from $\mathbb{P}^n$.

Given a Riemann surface X we can associate to it its field of meromorphic functions and any curve C which is the image of a holomorphic embedding of X in $\mathbb{P}^n$.

From a field K we can recover the curve or the Riemann surface as a point set by considering the valuation rings R, such that $\mathbb{C} \subset R \subset K$.

Example 4. Fix the genus $g = 0$.

Then there is only one curve with genus 0. It is the Riemann sphere $\mathbb{C} \cup \{\infty\}$ as a Riemann surface, or as an algebraic curve we think of it as projective space $\mathbb{P}^1$, or as a function field it is $\mathbb{C}(X)$.

Example 5. Let $g = 1$. We get the theory of elliptic curves. As a Riemann surface we get $\mathbb{C}/\Lambda$ where Λ is a lattice, and we can take $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$ where τ is in the upper half plane. As a curve, this is any non-singular plane cubic curve $f(x, y, z) = 0$ where f is homogeneous and degree three, and some partial of f does not vanish at each root. The function field is just $\mathbb{C}(X, \sqrt{f(X)})$ where f is degree three with distinct roots.

To get from one to the other we can use complex uniformization: we have the Weierstrass which takes a lattice $\mathbb{C}/\Lambda$ to $\mathbb{P}^2$ by $z \mapsto (z, \wp(z), \wp'(z))$ for $z \notin \Lambda$ and for $z \in \Lambda$ we map $z \mapsto (0, 0, 1)$. Then conversely to go back to the Riemann surface we can take the map

$$(x, y) \mapsto \int_{(x_0,y_0)}^{(x,y)} \frac{dx}{y}.$$

This is well defined up to periods which are inside the lattice Λ, so this defines a map $C \mapsto \mathbb{C}/\Lambda$.

For the exercises you may need the following proposition:

Proposition 1. *Let X be a Riemann surface and $h : X \to X$ be an automorphism of finite order. That is, $h^n = id$ for some $n \in \mathbb{N}$. Then there is a Riemann surface Y and a morphism $f : X \to Y$ such that for any pair of elements $x, x' \in X$ with $f(x) = f(x')$ there is a unique m such that $h^m(x) = x'$. The ramification points of f are those fixed by h^k for some $k < n$.*

In other words, Y is the quotient of X by the equivalence relation $x \sim x'$ if $x = h^m x'$ for some m.

In other other words, if G is a finite group of automorphisms of some Riemann surface X, then $X/G = Y$ is well defined and acts simply transitively on the fibers of $f : X \to X/G$.

Exercise 2 (Good Straightforward Practice)**.** Use Riemann Hurwitz to compute the genus of the following Riemann surfaces:

(a) $y^2 = (x^4 - 1)(x - 1)$

(b) $y^3 = (x^2 - 1)/(x^2 + 1)$

(c) $y^n = 1 - x^n$

Exercise 3 (Fun geometry!)**.** Let E denote $\mathbb{C}/\Lambda$ where $\Lambda = \mathbb{Z} + i\mathbb{Z}$. Multiplication by i sends $\Lambda \mapsto \Lambda$ rotating by ninety degrees, so we can mod out by the equivalence $x \sim ix$ to obtain a map $E \to E/\sim$. (You may assume that $E/\sim$ is a Riemann surface.) Find the ramification points of f. There should be some of order 4 and some of order 2.

Exercise 4 (Interesting, Hard)**.** Show that if X is a compact Riemann surface of genus $g \geq 2$, then there are at most $84(g - 1)$ automorphisms of X.

This is quite hard, but interesting. First of all, you may assume that X has only finitely many automorphisms, as the proof of that is beyond the scope of this material. From here, you might want to consider some quotients of X by principally generated subgroups of the automorphism group and apply Riemann Hurwitz. Then there is some casework with numbers.

Exercise 5 (Fun geometry!)**.** Let C be the Klein quartic $x^3y + y^3z + z^3x = 0$ in $\mathbb{P}^2$.

(a) Show that C has at least 168 automorphisms.

(b) Conclude that if the genus of C is 3, then these are all the automorphisms.

(c) Project C to $\mathbb{P}^1\mathbb{C}$ by mapping $[x : y : z]$ to $[x : y]$ when $z \neq 0$. Extend this map to $z = 0$.

(d) Find the degree of this map and the ramification points.

(e) Show that the genus of the Klein quartic is 3, and conclude that it has exactly 168 automorphisms.

Exercise 6 (Unimportant, but fun if you like classical algebraic geometry)**.** This exercise is about dual curves.

(a) Show that the dual of a smooth conic is a smooth conic.

(b) In general, is the dual of a smooth degree d curve also smooth?

(c) Compute the dual curve of the hyperbola $xy - 1$.