How to Prove the Hodge Conjecture

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Introduction

In many fields, such as topology, symplectic/complex geometry, algebraic geometry, an important strategy for studying spaces is to study their *invariants*. These are quantities or objects attached to a space such that if two spaces are equivalent in whatever sense one is interested (isomorphic, homeomorphic, homotopy equivalent, etc.), then their invariants are all equal.

Probably the richest invariants studied for such spaces are the *(co)homology groups*. These are certain abelian groups (or sometimes vector spaces) that encode a lot of deep information about the space.

Some of the most famous problems in modern mathematics were resolved by cohomology calculations.

Homology first steps

The story typically begins with the *singular homology* of a space X, which studies the following sequence

$$
\begin{aligned}
&\xleftarrow{\partial_{n+1}} \mathbb{Q}\{\text{dimension } n+1 \text{ subspaces}\} \xleftarrow{\partial_{n+2}} \cdots \\
&\xleftarrow{\partial_n} \mathbb{Q}\{\text{dimension } n \text{ subspaces}\} \\
\cdots &\xleftarrow{\partial_{n-1}} \mathbb{Q}\{\text{dimension } n-1 \text{ subspaces}\}
\end{aligned}
$$

where

- $\mathbb{Q}\{\dots\}$ means the vector space over $\mathbb{Q}$ with a basis vector for each element of the set, and
- ∂_n is the *boundary operator*. For example

$$\partial_1(\text{curve from } a \text{ to } b) = [b] - [a].$$

Homology first steps

Given the above sequence one constructs the *nth homology group* as

$$H_n(X, \mathbb{Q}) = \frac{\ker(\partial_n)}{\operatorname{im}(\partial_{n+1})},$$

i.e. the group of dimension n subspaces with trivial boundary (called *cycles*) up to equivalence by boundaries of dimension $n+1$ subspaces.

Exercise

Show that

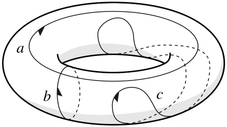
$$H_0(X, \mathbb{Q}) = \mathbb{Q}^m$$

where m *is the number of path-connected components of* X*.*

Homology of the torus

Example

Consider the torus $T \cong \mathbb{R}^2/\mathbb{Z}^2$, pictured with some cycles below.



To compute $H_1(T, \mathbb{Q})$ we need cycles to have trivial boundary (i.e. loops rather than curves) and we set the boundarys of any 2d subspaces to zero.

From this one can prove that $H_1(T, \mathbb{Q}) = \mathbb{Q}\{a, b\} \cong \mathbb{Q}^2$.

Cute application

One can compute that for the usual Euclidean space we have

$$H_k(\mathbb{R}^n \setminus \{0\}, \mathbb{Q}) = \begin{cases} \mathbb{Q}, & \text{if } k = 0, n \\ 0, & \text{else.} \end{cases}$$

Corollary

If $m \neq n$, then $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic.

Proof.

Since homology is an invariant, if they were homeomorphic these should be equal. □

This "obvious" fact is curiously difficult to prove for general m and n otherwise.

Cohomology first steps

So in general for a co̲ho̲mlogy theory we want a sequence

$$\mathbb{Q}\{\text{dimension } n+1 \text{ data on } X\} \xrightarrow{d_{n+1}} \cdots$$

$$\mathbb{Q}\{\text{dimension } n \text{ data on } X\} \xrightarrow{d_n}$$

$$\cdots \xrightarrow{d_{n-2}} \mathbb{Q}\{\text{dimension } n-1 \text{ data on } X\} \xrightarrow{d_{n-1}}$$

from which we'd set the cohomolgy to be $H^n(X) = \dfrac{\ker(d_n)}{\mathrm{im}(d_{n-1})}$.

Example

If we take $\{$ dim n data $\}$ to be the set of functions on $\mathbb{Q}\{$ dim n subspaces $\}$, i.e. the *dual*, with $d_n(f) = f \circ \partial_{n+1}$, we get the *singular cohomology groups* $H^n(X, \mathbb{Q})$.

De Rham Cohomology first steps

For the purpose of today's talk we'll be focusing on a cohomology theory called *de Rham cohomology*. Here our dim n data on X is the set

$$\Omega^n(X) = \{ \text{ differential } k\text{-forms } \}$$

and the map d_n is the *exterior derivative*.

Differential forms are objects like $f(x,y)dx$, $g(x,y)dx \wedge dy$, etc. In the complex case one has forms such as $h(z)dz \wedge d\bar{z}$, where $dz = dx + idy$.
(Note that $dx \wedge dx = 0$ and $dx \wedge dy = -dy \wedge dx$.)
The exterior derivative takes the total differential of the function, for example

$$d_1(f(x,y)dx) = (f_x dx + f_y dy) \wedge dx = f_y dy \wedge dx = -f_y dx \wedge dy.$$

Complex de Rham cohomology

The sequence $(\Omega^n(X), d_n)$ defines the de Rham cohomology groups $H^n_{\mathrm{dR}}(X)$. These are vector spaces over whatever field X is defined.

In the complex case if the variables on X are $z_1, \dots, z_n$, then we have a dichotomy between the *holomorphic* differentials dz_i and the *antiholomorphic* differentials $d\overline{z}_j$. This induces a decomposition (called the *Hodge decomposition*)

$$H^k_{\mathrm{dR}}(X) = \bigoplus_{i+j=k} H^{i,j}(X),$$

where $H^{i,j}(X)$ consists of classes of the form

$$dz_1 \wedge \dots \wedge dz_i \wedge d\overline{z}_{i+1} \wedge \dots \wedge d\overline{z}_{i+j}.$$

The Cycle Class Map

Differential forms often arise for integrating over the subspaces of X. Consider the $\mathbb{Q}$-vector space $C^r(X)_{\mathbb{Q}}$ of subspaces $Z \subseteq X$ of *codimension* r (think "cut out by r equations"). We get a map to the *dual* of de Rham cohomology

$$\begin{aligned} C^r(X) &\longrightarrow H^{2(n-r)}_{\mathrm{dR}}(X)^\vee \\ Z &\longmapsto \left(\omega \mapsto \int_Z \omega \right). \end{aligned}$$

The trick to get a map to cohomology itself is to use a big theorem called *Poincare duality*, which furnishes us with an isomorphism $H^{2(n-r)}_{\mathrm{dR}}(X)^\vee \xrightarrow{\sim} H^{2r}_{\mathrm{dR}}(X)$.

Hence we get the *cycle class map* $c : C^r(X) \to H^{2r}_{\mathrm{dR}}(X)$.

The Hodge Conjecture

Some important notes about the cycle class map:

- Recall we also had a duality between subspaces and cohomology when we discussed singular cohomology, and indeed the cycle class map does land in $H^{2r}(X, \mathbb{Q}) \subseteq H^{2r}_{\mathrm{dR}}(X)$.
- In terms of our Hodge decomposition, the cycle class map always lands in $H^{r,r}(X)$.

Conjecture (Hodge Conjecture)

Every class in $H^{2r}(X, \mathbb{Q}) \cap H^{r,r}(X)$ is the image of some $Z \in C^r(X)_{\mathbb{Q}}$ under the cycle class map.

Elliptic Curves

Let's actually get to an example!

Complex elliptic curves are quotient spaces of the form $E = \mathbb{C}/\Lambda$, where Λ is some lattice in $\mathbb{C}$. (In fact we can always take $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\tau$, with $\mathrm{Im}(\tau) > 0$). Topologically then, these are just tori as we saw before! This allows us to compute the homology, and it is possible to also compute the de Rham cohomology as follows:

$$H_0(E, \mathbb{Q}) = \mathbb{Q} \quad H_1(E, \mathbb{Q}) = \mathbb{Q}\{\gamma, \gamma'\} \cong \mathbb{Q}^2 \quad H_2(E, \mathbb{Q}) = \mathbb{Q}$$
$$H^0_{\mathrm{dR}}(E) = \mathbb{C} \quad H^1_{\mathrm{dR}}(E) = \mathbb{C}\{dz, d\bar{z}\} \cong \mathbb{C}^2 \quad H^2_{\mathrm{dR}}(E) = \mathbb{C}.$$

Since the only subspaces of E of positive codimension are just points, they are not so interesting from the perspective of the Hodge conjecture.

A Power of an Elliptic Curve

Consider instead the space $E \times E$, which is now two dimensional over $\mathbb{C}$. The space of codimension 1 subspaces (up to "algebraic equivalence") is denoted $\mathrm{NS}(E \times E)_{\mathbb{Q}}$, and importantly on this space the cycle class map c is *injective*.

Since these are just curves on $E \times E$ they are either "vertical", "horizontal", or "diagonal". In the vertical and horizontal case they just descend to cycles on the first and second copies of E. In the diagonal case we actually get the graph of a function $E \to E$. Thus we get:

$$\begin{aligned} \mathrm{NS}(E \times E)_{\mathbb{Q}} &= \mathrm{NS}(E)_{\mathbb{Q}} \oplus \mathrm{NS}(E)_{\mathbb{Q}} \oplus \mathrm{End}(E)_{\mathbb{Q}} \\ &\cong \mathbb{Q} \oplus \mathbb{Q} \oplus \begin{cases} \mathbb{Q}^2, & \text{if } \tau + \overline{\tau}, \tau\overline{\tau} \in \mathbb{Q} \\ \mathbb{Q}, & \text{else.} \end{cases} \end{aligned}$$

The Hodge Conjecture for $E \times E$

We now have an *injective* cycle class map of $\mathbb{Q}$-vector spaces

$$c : \mathrm{NS}(E \times E)_{\mathbb{Q}} \longrightarrow H^2(E \times E, \mathbb{Q}) \cap H^{1,1}(E \times E).$$

Thus if our goal is to prove the $r = 1$ Hodge conjecture for $X = E \times E$ we are reduced to proving the following:

Proposition

$$\dim H^2(E \times E, \mathbb{Q}) \cap H^{1,1}(E \times E) = \begin{cases} 4, & \textit{if } \tau + \overline{\tau}, \tau\overline{\tau} \in \mathbb{Q} \\ 3, & \textit{else.} \end{cases}$$

The final ingredient we need is how to view the singular cohomology $H^2(E \times E, \mathbb{Q})$ inside the de Rham cohomology.

The Proof

Write δ, δ' for the basis of $H^1(E, \mathbb{Q})$ dual to the basis γ, γ' of $H_1(E, \mathbb{Q})$. Then we have

$$dz = \delta + \tau\delta' \qquad \text{and} \qquad d\overline{z} = \delta + \overline{\tau}\delta',$$

where τ is as in $E = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$.
Writing z_1, z_2 for the coordinates on each copy of E, we have

$$H^{1,1}(E \times E) = \mathbb{C}\{dz_1 \wedge d\overline{z}_1, dz_1 \wedge d\overline{z}_2, d\overline{z}_1 \wedge dz_2, dz_2 \wedge d\overline{z}_2\}.$$

The strategy is then to expand linear combinations of these in terms of $\delta_1, \delta_1', \delta_2, \delta_2'$ and find which ones are possible over $\mathbb{Q}$.

The Proof Continues

Since we want rational classes, in particular they must be real (i.e. preserved by complex conjugation) so it is sufficient to consider the classes

$$\begin{aligned}
&a(dz_1 \wedge d\overline{z}_2) + \overline{a}(d\overline{z}_1 \wedge dz_2) \\
&\qquad = a(\delta_1 + \tau\delta_1') \wedge (\delta_2 + \overline{\tau}\delta_2') + \overline{a}(\delta_1 + \overline{\tau}\delta_1') \wedge (\delta_2 + \tau\delta_2') \\
&\qquad = (a + \overline{a})\delta_1 \wedge \delta_2 + (a\overline{\tau} + \overline{a}\tau)\delta_1 \wedge \delta_2' + (a\tau + \overline{a\tau})\delta_1' \wedge \delta_2 \\
&\qquad\qquad + (a + \overline{a})(\tau\overline{\tau})\delta_1' \wedge \delta_2' \\
b_i(dz_i \wedge d\overline{z}_i) &= b_i(\delta_i + \tau\delta_i') \wedge (\delta_i + \overline{\tau}\delta_i') \\
&= b_i(\overline{\tau}\delta_i \wedge \delta_i' + \tau\delta_i' \wedge \delta_i) = b_i(\overline{\tau} - \tau)\delta_i \wedge \delta_i'
\end{aligned}$$

The Proof Never Ends

This gives us the conditions

$$a+\overline{a}\in\mathbb{Q},\quad a\overline{\tau}+\overline{a}\tau\in\mathbb{Q},\quad a\tau+\overline{a\tau}\in\mathbb{Q},\quad (a+\overline{a})(\tau\overline{\tau})\in\mathbb{Q},$$

and $b_i(\overline{\tau}-\tau)\in\mathbb{Q}$. This last condition gives two $\mathbb{Q}$-degrees of freedom, one each for b_1 and b_2. (If you're careful you can see these ones correspond to the two factors of $\mathrm{NS}(E)_{\mathbb{Q}}$.)

The first and fourth condition above tell us either $a+\overline{a}=0$, or $\tau\overline{\tau}\in\mathbb{Q}$. Adding the second and third gives $(a+\overline{a})(\tau+\overline{\tau})\in\mathbb{Q}$ so again either $a+\overline{a}=0$, or $\tau+\overline{\tau}\in\mathbb{Q}$. The difference of the second and third gives the condition $(a-\overline{a})(\tau-\overline{\tau})\in\mathbb{Q}$.

Oh Wait Yes It Does

Hence we get one $\mathbb{Q}$-degree of freedom for $a - \overline{a}$, and a second for $a + \overline{a}$ if $\tau + \overline{\tau}, \tau\overline{\tau} \in \mathbb{Q}$.

Putting all the above together gives

$$\dim H^2(E \times E, \mathbb{Q}) \cap H^{1,1}(E \times E) = \begin{cases} 4, & \text{if } \tau + \overline{\tau}, \tau\overline{\tau} \in \mathbb{Q} \\ 3, & \text{else,} \end{cases}$$

as desired! □

Final Remarks

While this computation is good fun, it's unfortunately not a "new" result, as it were. The Hodge conjecture is open in general, but is known in the following cases:

- Any X in the case $r = 1$, known as the *Lefschetz* $(1,1)$-*theorem*.
- Any abelian variety X.
- Probably some other cases I dunno.

Thanks for listening!