Problem Set 1 - Solutions

2019 Math Boot Camp for the Political and Social Sciences

Some practice

1. Consider the sets $A = \{1, 3, 5, 7, 9\}$ and $B = \{1, 2, 3, 4, 5\}$.
   (a) $A \cap B = \{1, 3, 5\}$.
   (b) $A \cup B = \{1, 2, 3, 4, 5, 7, 9\}$.
   (c) The mean of A is $\frac{1}{5}(1 + 3 + 5 + 7 + 9) = 5$.
   (d) The mean of B is $\frac{1}{5}(1 + 2 + 3 + 4 + 5) = 3$.
2. $\frac{128}{24} = \frac{16}{3}$ and $\frac{24}{128} = \frac{3}{16}$, so
   (a) $\frac{128}{24} \times \frac{24}{128} = \frac{16}{3} \times \frac{3}{16} = 1$
   (b) $\frac{16}{3} + \frac{3}{16} = \frac{256}{48} + \frac{9}{48} = \frac{265}{48}$
3. $3t - 5t + 4 = 2$, so collecting like terms we get $-2t = -2$. Thus $t = 1$.
4. $(2 - b)(b + 3) = 0$ so either $2 - b = 0$ or $b + 3 = 0$. Thus the solution is $b = 2$ or $b = -3$.
5. $(2x + 3y)^2 = (2x + 3y)(2x + 3y) = 4x^2 + 6xy + 6xy + 9y^2 = 4x^2 + 12xy + 9y^2$.
6. $x^2 + 5x + 6 = (x + 2)(x + 3)$.
7. $\dfrac{x^2 + x}{xy + x + y + 1} = \dfrac{x(x + 1)}{(x + 1)(y + 1)} = \dfrac{x}{y + 1}$.

Deeper Thinking

1.
$$A \cap B = \{\text{the set of all multiples of } 6\}$$
Some good examples to try when searching for a generalisation would be multiples of 3 and 5. Then try multiples of 2 and 4. Any conjectures?

2.
$$\begin{aligned}
x^2 - y^2 &= (x - y)(x + y) \\
x^3 - y^3 &= (x - y)(x^2 + xy + y^2) \\
x^4 - y^4 &= (x - y)(x^3 + x^2y + xy^2 + y^3) = (x - y)(x + y)(x^2 + y^2)
\end{aligned}$$
So there's always the factor of $(x - y)$, but sometimes there's more! Any conjectures?

3. Start with the quadratic equation $ax^2 + bx + c = 0$. Then

$$\begin{aligned}
ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) \\
&= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} + \frac{c}{a} - \frac{b^2}{4a^2}\right) \\
&= a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right) \\
&= a\left(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right)
\end{aligned}$$

Thus we see that we have solutions $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

4. Assume that it was possible, so $\sqrt{2} = \frac{p}{q}$, with p, q whole numbers *with no common factors* (this turns out to be crucial). Then

$$\begin{aligned}
\sqrt{2} &= \frac{p}{q} \\
2 &= \frac{p^2}{q^2} \\
2q^2 &= p^2
\end{aligned}$$

This shows p is even, so we can write $p = 2k$. Then

$$\begin{aligned}
2q^2 &= (2k)^2 \\
2q^2 &= 4k^2 \\
q^2 &= 2k^2
\end{aligned}$$

This shows q is even, but p and q can't both be even since then they have a common factor of 2! Thus we have a contradiction and it must be impossible.