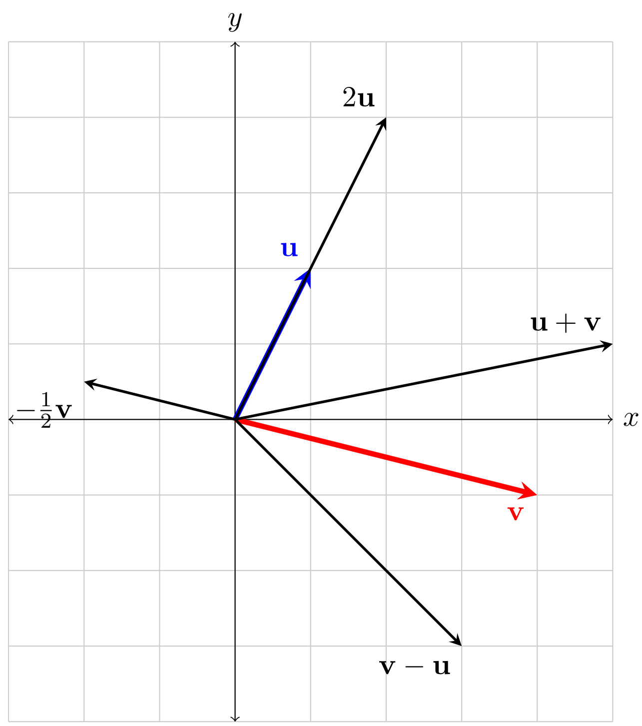
Problem Set 3 - Solutions

2019 Math Boot Camp for the Political and Social Sciences

Some practice

1.



2. (a) $||\mathbf{u}|| = \sqrt{1^2+2^2} = \sqrt{5}$ and $||\mathbf{v}|| = \sqrt{4^2+(-1)^2} = \sqrt{17}$.
   (b) $||\mathbf{u}+\mathbf{v}|| = \sqrt{5^2+1^2} = \sqrt{26}$ and one can check $\sqrt{26} < \sqrt{5}+\sqrt{17}$.
   (c) $|\mathbf{u}\cdot\mathbf{v}| = |1\times 4 + 2\times -1| = |4-2| = 2$ and one can check $2 < \sqrt{5}\sqrt{17}$.

3. (a) $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 2 & 2 \end{pmatrix}$
   (b) $\det \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = 1\times 1 - 2\times 0 = 1$ and $\det \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} = 2\times 1 - 3\times 2 = -4$.
   (c) $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}^{-1} = \frac{1}{-4}\begin{pmatrix} 1 & -3 \\ -2 & 2 \end{pmatrix}$.

4. (a) We can rearrange the second equation to read $y = 2x - 3$ and substitute this into the first equation to get

$$\begin{aligned} 2x + 3(2x-3) &= 7 \\ 8x - 9 &= 7 \\ 8x &= 16 \end{aligned}$$

so $x = 2$ and thus $y = 2 \times 2 - 3 = 1$.

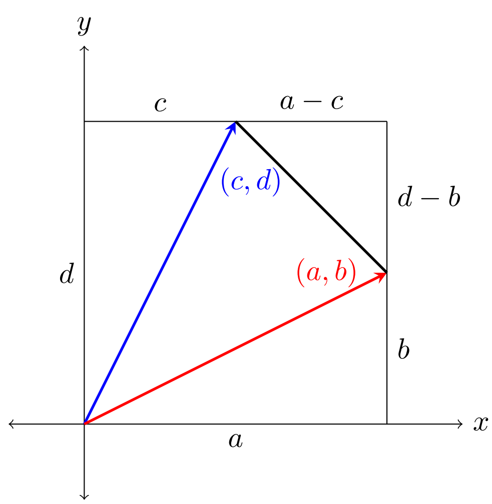
(b) As a matrix equation we have

$$\begin{aligned} \begin{pmatrix} 2 & 3 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 7 \\ 6 \end{pmatrix} \\ \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 2 & 3 \\ 4 & -2 \end{pmatrix}^{-1} \begin{pmatrix} 7 \\ 6 \end{pmatrix} \\ \begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{-16} \begin{pmatrix} -2 & -3 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} 7 \\ 6 \end{pmatrix} \end{aligned}$$

Thus $x = \frac{1}{-16}(-2 \times 7 + -3 \times 6) = \frac{-32}{-16} = 2$ and $y = \frac{1}{-16}(-4 \times 7 + 2 \times 6) = \frac{-16}{-16} = 1$.

Deeper Thinking

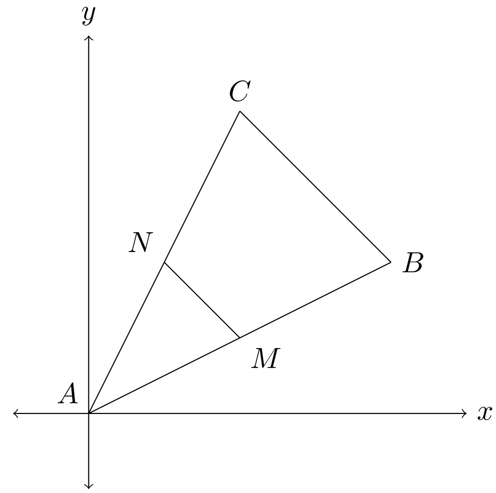
1. We'll do the case where $ad - bc$ is positive (so we don't need to worry about the absolute values) using the following picture



So we see the area of the inner triangle is the area of the outer rectangle minus the three right triangles, giving

$$\begin{aligned} \text{Area} &= ad - \frac{1}{2}ab - \frac{1}{2}(a-c)(d-b) - \frac{1}{2}cd \\ &= ad - \frac{1}{2}ab - \frac{1}{2}ad + \frac{1}{2}ab + \frac{1}{2}cd - \frac{1}{2}bc - \frac{1}{2}cd \\ &= \frac{1}{2}ad - \frac{1}{2}bc = \frac{1}{2}(ad - bc). \end{aligned}$$

2. In pictures this setup looks like



If we call the side AB the vector $\mathbf{u}$ and AC the vector $\mathbf{v}$ we see

$$BC = \mathbf{v} - \mathbf{u}$$
$$AM = \frac{1}{2}\mathbf{u}$$
$$AN = \frac{1}{2}\mathbf{v}$$

therefore $MN = \frac{1}{2}\mathbf{v} - \frac{1}{2}\mathbf{u} = \frac{1}{2}BC$.

3. Consider the vectors $(\sqrt{a_1}, \sqrt{a_2}, \dots, \sqrt{a_n})$ and $\left(\frac{1}{\sqrt{a_1}}, \frac{1}{\sqrt{a_2}}, \dots, \frac{1}{\sqrt{a_n}}\right)$. Then the Cauchy-Schwartz inequality gives us

$$\sqrt{a_1 + a_2 + \dots + a_n}\sqrt{\frac{1}{a_1} + \frac{1}{a_2} + \dots \frac{1}{a_n}} \geq \frac{\sqrt{a_1}}{\sqrt{a_1}} + \frac{\sqrt{a_2}}{\sqrt{a_2}} + \dots + \frac{\sqrt{a_n}}{\sqrt{a_n}} = 1 + 1 + \dots + 1 = n$$

Squaring both sides gives

$$(a_1 + \dots + a_n)\left(\frac{1}{a_1} + \dots + \frac{1}{a_n}\right) \geq n^2.$$