Problem Set 4 - Solutions

2019 Math Boot Camp for the Political and Social Sciences

Some practice

1.

$$AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1\times 2+2\times 1 & 1\times 2+2\times 2 \\ 3\times 2+4\times 1 & 3\times 2+4\times 2 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 10 & 14 \end{pmatrix}$$

$$BA = \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 2\times 1+2\times 3 & 2\times 2+2\times 4 \\ 1\times 1+2\times 3 & 1\times 2+2\times 4 \end{pmatrix} = \begin{pmatrix} 8 & 12 \\ 7 & 10 \end{pmatrix}$$

2.

$$AA^{-1} = \begin{pmatrix} 5 & 3 \\ 4 & 3 \end{pmatrix}\frac{1}{3}\begin{pmatrix} 3 & -3 \\ -4 & 5 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 5\times 3+3\times -4 & 5\times -3+3\times 5 \\ 4\times 3+3\times -4 & 4\times -3+3\times 5 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

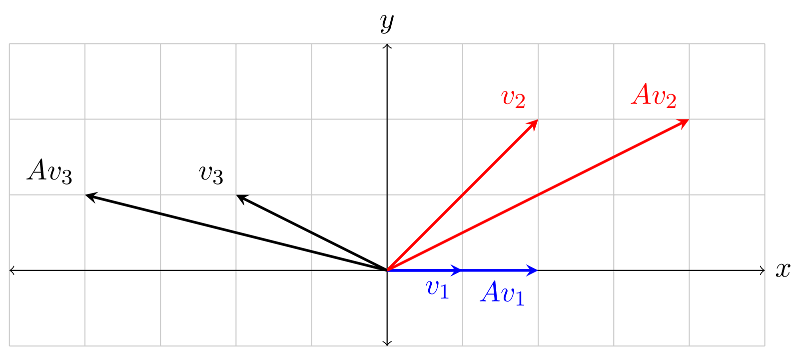
$$A^{-1}A = \frac{1}{3}\begin{pmatrix} 3 & -3 \\ -4 & 5 \end{pmatrix}\begin{pmatrix} 5 & 3 \\ 4 & 3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 3\times 5+-3\times 4 & 3\times 3+-3\times 3 \\ -4\times 5+5\times 4 & -4\times 3+5\times 3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

3. (a) $Av = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1\times 1+2\times -1 \\ 2\times 1+1\times -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

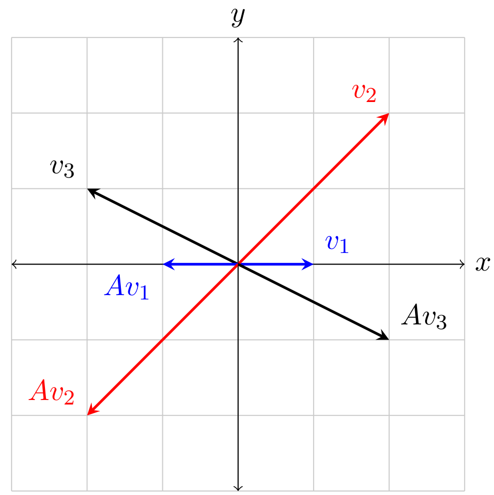
   (b) $Av = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 3\times 0+-1\times 3 \\ 1\times 0+1\times 3 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$

   (c) $Av = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0\times 2+1\times 1 \\ 1\times 2+2\times 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

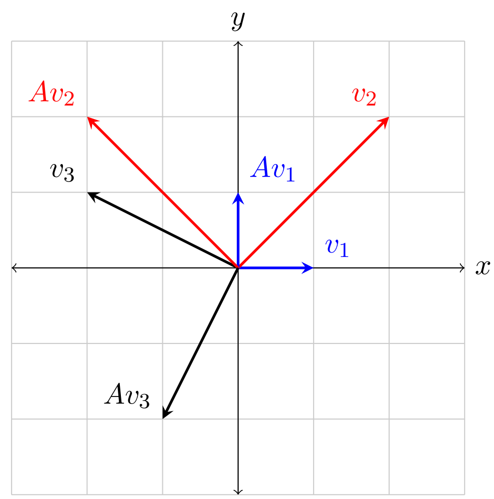
4. (a) The matrix $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ corresponds to dilating by a factor of 2 along the x-axis, as can be seen in the diagram



(b) The matrix $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ corresponds to reflecting across the origin (or equivalently a 180° rotation), as can be seen in the diagram



(c) The matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ corresponds to a 90° counterclockwise rotation, as can be seen in the diagram



Deeper Thinking

1. For $v = \begin{pmatrix} x \\ y \end{pmatrix}$ the equation $Av = v$ gives equations

$$\begin{aligned} 0.3x + 0.6y &= x \\ 0.7x + 0.4y &= y \end{aligned}$$

In fact these equations are exactly the same, so we just focus on the first, $0.7x = 0.6y$. Combined with our condition $x + y = 13$ we get $x = 6$ and $y = 7$, so $v = \begin{pmatrix} 6 \\ 7 \end{pmatrix}$.

I'll leave you to compute $v_1 = Av_0$, $v_2 = Av_1$, etc and see what you notice.

2. What conditions on $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ mean that for *every* B we have $AB = BA$? We can check at least if $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ then it will satisfy $AB = BA$ for every B. In fact we can see that these are the *only* A that work by considering the following two cases for B:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & 0 \\ d & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}$$

The fact that we need $AB = BA$ on the left tell us $a = d$ and $c = 0$. The fact that we need $AB = BA$ on the right tell us $a = d$ and $b = 0$.

So we get that the only A that can work are $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$.