Problem Set 5 - Solutions

2019 Math Boot Camp for the Political and Social Sciences

Some practice

1. We'll show off the approach using the definition here

   (a) $f'(x) = \lim_{h \to 0} \frac{5 - 5}{h} = 0$

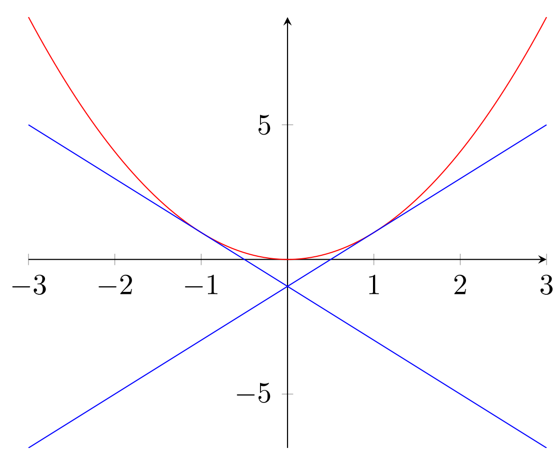
   (b) $f'(x) = \lim_{h \to 0} \frac{x + h - x}{h} = \lim_{h \to 0} \frac{h}{h} = 1$

   (c) $f'(x) = \lim_{h \to 0} \frac{(x + h + 1)^2 + 1 - (x + 1)^2 - 1}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + 2x + 2h + 1 - x^2 - 2x - 1}{h}$
   $= \lim_{h \to 0} \frac{2xh + 2h + h^2}{h} = \lim_{h \to 0} 2x + 2 + h = 2x + 2$

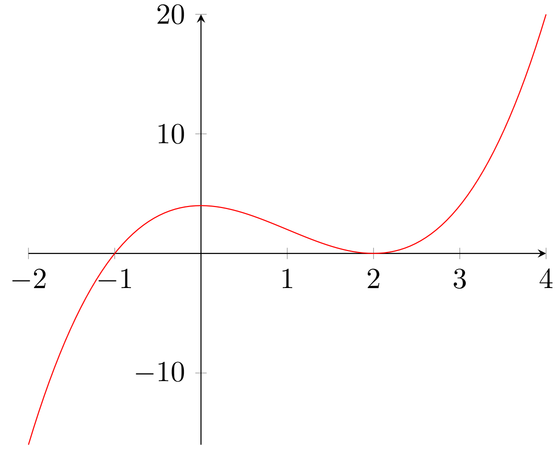
   (d) $f'(x) = \lim_{h \to 0} \frac{2(x + h) - (x + h)^2 - 2x + x^2}{h} = \lim_{h \to 0} \frac{2x + 2h - x^2 - 2xh - h^2 - 2x + x^2}{h}$
   $= \lim_{h \to 0} \frac{2h - 2xh - h^2}{h} = \lim_{h \to 0} 2 - 2x - h = 2 - 2x$

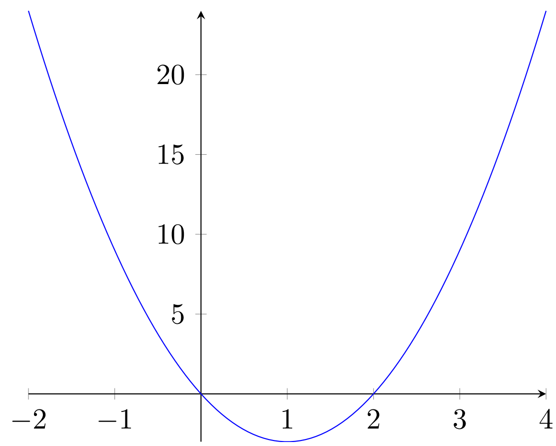
   (e) $f'(x) = \lim_{h \to 0} \frac{(x + h - 3)(x + h + 2) + (x + h) - (x - 3)(x + 2) - x}{h}$
   $= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x - h - 6 + x + h - x^2 + x + 6 - x}{h}$
   $= \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} 2x + h = 2x$

2.



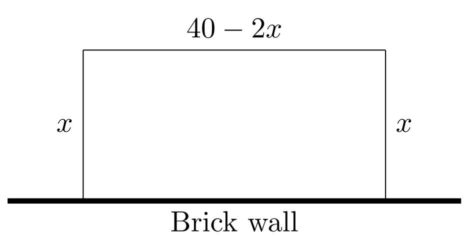
3.





The x-intercepts of $f'(x)$ are $x = 0$ and $x = 2$, which correspond to the local maximum and minimum of $f(x)$.

4. We let the two sides touching the brick wall be length x. This setup looks like



So the area is $A(x) = (40 - 2x)x = 40x - 2x^2$. To find the maximum, we compute $A'(x) = 40 - 4x$. Setting $A'(x) = 0$ we get $x = 10$. Thus the maximum area is $A(10) = 40 \times 10 - 2 \times 10^2 = 400 - 200 = 200$.

Deeper Thinking

1. The idea is that the second derivative $f''(x)$ describes the *rate of change of the derivative* $f'(x)$. So when $f'(x) = 0$:

   - If $f''(x) > 0$, that means the derivative was increasing. Thus it must have gone from negative to zero to positive. Thus it must be a minimum.
   - If $f''(x) > 0$, that means the derivative was decreasing. Thus it must have gone from positive to zero to negative. Thus it must be a maximum.

2. We find polynomials satisfying just the first condition, then the first two, then the first three, etc. This gives us

$$\begin{aligned}
f(x) &= 1 \\
f(x) &= 1 + x \\
f(x) &= 1 + x + \frac{x^2}{2} \\
f(x) &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} \\
f(x) &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \dots
\end{aligned}$$

   I'll leave you to plot these and see if you notice anything as you add higher and higher powers...

3. We find polynomials satisfying just the first condition, then the first two, then the first three, etc. This gives us

$$\begin{aligned}
f(x) &= 0 \\
f(x) &= x \\
f(x) &= x - \frac{x^3}{3} \\
f(x) &= x - \frac{x^3}{3} + \frac{x^5}{5} \\
f(x) &= x - \frac{x^3}{3} + \frac{x^5}{5} + \dots
\end{aligned}$$

   I'll leave you to plot these and see if you notice anything as you add higher and higher powers...