Problem Set 6 - Solutions

2019 Math Boot Camp for the Political and Social Sciences

Some practice

1. (a) $f'(x) = -\frac{1}{x^2} + \frac{1}{2\sqrt{x}}$

   (b) $f'(x) = 2e^x - \frac{5}{x}$

   (c) $f'(x) = \cos(x) - \sin(x)$

2. (a) We set $u(x) = \sin(x)$ so $f(u) = e^u$. Thus

$$f'(x) = u'(x)f'(u) = \cos(x)e^u = \cos(x)e^{\sin(x)}$$

   (b) The product rule gives us

$$f'(x) = (2x + \frac{3}{2\sqrt{x}})\log(x) + \frac{x^2 + 3\sqrt{x}}{x}$$

   (c) We set $u(x) = \cos(x)$ so $f(u) = \log(u)$. Thus

$$f'(x) = u'(x)f'(u) = -\sin(x)\frac{1}{u} = -\frac{\sin(x)}{\cos x} = -\tan(x)$$

3. (a) To find $P'(t)$ we set $u(t) = 8t - t^2$ so $P(u) = e^u$. Thus

$$P'(t) = u'(t)P'(u) = (8-2t)e^u = (8-2t)e^{8t-t^2}$$

   Then $P'(t) = 0$ and the maximum is reached when $t = 4$ since e^{8t-t^2} is never zero.

   (b) That population is $P(4) = e^{8\times 4 - 4^2} = e^{16} \approx 8,886,110$.

   (c) As noted above the exponential e^{8t-t^2} is never zero (though as t gets really massive it gets really close, so given that we usually work with whole number populations it would probably actually get to zero).

Deeper Thinking

1. We can compute directly here

$$\frac{d}{dx}\cos(x) = \frac{d}{dx}\frac{1}{2}(e^{ix} + e^{-ix}) = \frac{1}{2}(ie^{ix} - ie^{-ix}) = -\frac{1}{2i}(e^{ix} - e^{-ix}) = -\sin(x)$$

   and

$$\frac{d}{dx}\sin(x) = \frac{d}{dx}\frac{1}{2i}(e^{ix} - e^{-ix}) = \frac{1}{2i}(ie^{ix} + ie^{-ix}) = \frac{1}{2}(e^{ix} + e^{-ix}) = \cos(x)$$

2. Since e^x is always it's own derivative, we always get the same thing as $f(0) = e^0 = 1$, thus

$$f(0) = 1, \quad f'(0) = 1, \quad f''(0) = 1, \quad f'''(0) = 1, \quad f''''(0) = 1, \quad f'''''(0) = 1, \quad \ldots$$

For $f(x) = \sin(x)$ we get $f'(x) = \cos(x)$, then $f''(x) = -\sin(x)$, then $f'''(x) = -\cos(x)$, then $f''''(x) = \sin(x)$ and the pattern repeats. Thus evaluating each of these at $x = 0$ we get the pattern

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = 0, \quad f'''(0) = -1, \quad f''''(0) = 0, \quad f'''''(0) = 1, \quad \ldots$$

How does this compare to the Deeper Thinking problems on Problem Set 5. What's going on here?

3. First up linearity

$$\begin{aligned}
\frac{d}{dx}(f(x) + g(x)) &= \lim_{h \to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h} \\
&= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\
&= f'(x) + g'(x)
\end{aligned}$$

Now the Product Rule

$$\begin{aligned}
\frac{d}{dx}(f(x)g(x)) &= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\
&= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x)}{h} + \lim_{h \to 0} \frac{f(x+h)g(x) - f(x)g(x)}{h} \\
&= \lim_{h \to 0} f(x+h) \frac{g(x+h) - g(x)}{h} + \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} g(x) \\
&= f(x)g'(x) + f'(x)g(x)
\end{aligned}$$

Finally the Chain Rule

$$\begin{aligned}
\frac{d}{dx} f(u(x)) &= \lim_{h \to 0} \frac{f(u(x+h)) - f(u(x))}{h} \\
&= \lim_{h \to 0} \frac{u(x+h) - u(x)}{h} \frac{f(u(x+h)) - f(u(x))}{u(x+h) - u(x)} \\
&= \lim_{h \to 0} \frac{u(x+h) - u(x)}{h} \lim_{u(x+h) \to u(x)} \frac{f(u(x+h)) - f(u(x))}{u(x+h) - u(x)} \\
&= u'(x) f'(u)
\end{aligned}$$

where in the last step we use a slightly altered expression for the limit, but it gives an exactly equivalent definition of derivative.