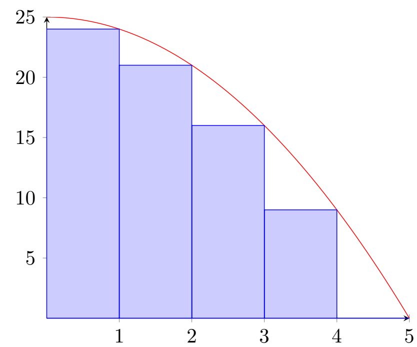
Problem Set 7 - Solutions

2019 Math Boot Camp for the Political and Social Sciences

Some practice

1.



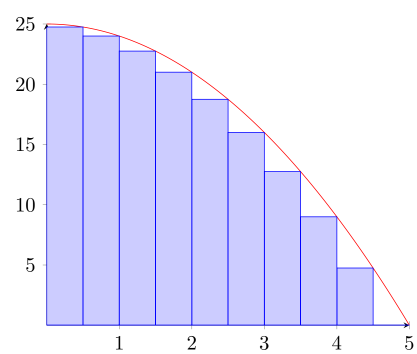
We have rectangles of width 1 and heights

$$f(1) = 25 - 1^2 = 24 \qquad f(2) = 25 - 2^2 = 21 \qquad f(3) = 25 - 3^2 = 16$$
$$f(4) = 25 - 4^2 = 9 \qquad f(5) = 25 - 5^2 = 0$$

Thus we have an approximate area of

$$1 \times 24 + 1 \times 21 + 1 \times 16 + 1 \times 9 + 1 \times 0 = 70.$$

Of course the calculation gets messier with smaller rectangles, but here at least is a picture for width $\frac{1}{2}$.



2. (a) $\int 3dx = 3x + c$, where c is an unknown constant.

(b) $\int x - 2x^2 dx = \frac{1}{2}x^2 - \frac{2}{3}x^3 + c$, where c is an unknown constant.

(c) $\int_0^5 (x+1)^2 dx = \int_0^5 x^2 + 2x + 1 dx = \frac{1}{3}x^3 + x^2 + x\Big|_0^5 = \frac{125}{3} + 25 + 5 = \frac{215}{3}$.

3. (a) $\int_1^7 3dx = 3x\Big|_1^7 = 21 - 3 = 18$.

   (b) $\int_0^3 2x + 2dx = x^2 + 2x\Big|_0^3 = 9 + 6 = 15$.

   (c) $\int_0^8 8x - x^2 dx = 4x^2 - \frac{1}{3}x^3\Big|_0^8 = 256 - \frac{512}{3} = \frac{256}{3}$.

4. The average is given by

$$\frac{1}{4-2}\int_2^4 x^3 - 3x + 2dx = \frac{1}{2}\left(\frac{1}{4}x^4 - \frac{3}{2}x^2 + 2x\Big|_2^4\right) = \frac{1}{2}(64 - 24 + 8 - 4 + 6 - 4) = 23.$$

5. The total distance travelled is given by

$$\int_0^4 1 + 2t + 3t^2 dt = t + t^2 + t^3\Big|_0^4 = 4 + 16 + 64 = 84.$$

Deeper Thinking

1. There are a few ways to compute this sum, but we'll give a tricky way using a formula we derived a few days ago.

   On Problem Set 1 we saw that

$$x^{n+1} - y^{n+1} = (x-y)(x^n + x^{n-1}y + \ldots + xy^{n-1} + y^n) = (x-y)\sum_{k=0}^{n} x^{n-k}y^k.$$

   Letting $x = 1$ and $y = \frac{1}{2}$ we get

$$1 - \frac{1}{2^{n+1}} = \left(1 - \frac{1}{2}\right)\sum_{k=0}^{n}\frac{1}{2^k} = \frac{1}{2}\sum_{k=0}^{n}\frac{1}{2^k}$$

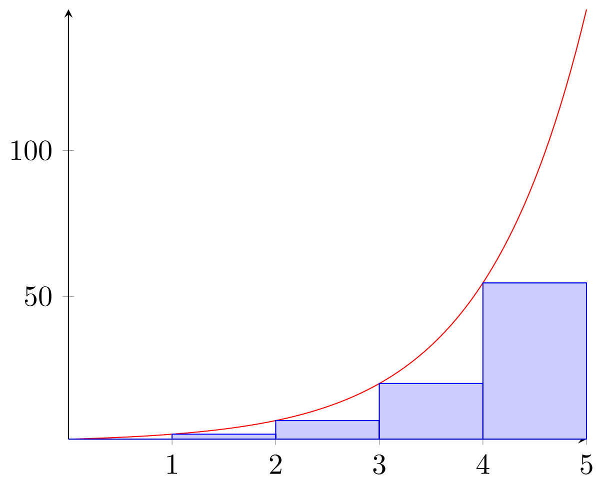
   and thus multiplying through by 2 we get

$$\sum_{k=0}^{n}\frac{1}{2^k} = 2 - \frac{1}{2^n}$$

   Then we can say

$$\sum_{k=0}^{\infty}\frac{1}{2^k} = \lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{2^k} = \lim_{n\to\infty}\left(2 - \frac{1}{2^n}\right) = 2.$$

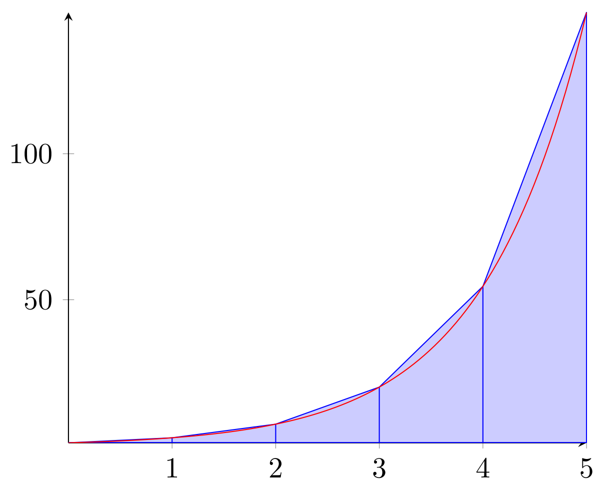
2.



The sum we are computing for this approximation is

$$1 + e + e^2 + e^3 + e^4 \approx 85.79$$

To upgrade to sloped lines we get



Since they are all of width 1, the area of each trapezium is $\frac{\text{short side} + \text{long side}}{2}$, so we compute

$$\frac{1+e}{2} + \frac{e+e^2}{2} + \frac{e^2+e^3}{2} + \frac{e^3+e^4}{2} + \frac{e^4+e^5}{2} \approx 159.5$$

I'll leave you to think about what the quadratic approximation might look like...