Problem Set 8 - Solutions

2019 Math Boot Camp for the Political and Social Sciences

Some practice

1. (a) $\int_0^3 e^x dx = e^x \Big|_0^3 = e^3 - e^0 = e^3 - 1.$

   (b) $\int 4\cos(x)dx = 4\sin(x) + c$, where c is an arbitrary constant.

   (c) $\int \frac{1}{x} - \sin(x)dx = \log(x) + \cos(x) + c$, where c is an arbitrary constant.

2. (a) Here we have $du = -dx$, thus we get

$$\int \frac{1}{1-x}dx = -\int \frac{1}{u}du = -\log(u) + c = -\log(1-x) + c,$$

   where c is an arbitrary constant.

   (b) Here we have $du = 2xdx$, thus we get

$$\int 2x\cos(x^2+1)dx = \int \cos(u)du = -\sin(u) + c = -\sin(x^2+1) + c,$$

   where c is an arbitrary constant.

   (c) Here we have $du = \frac{1}{x}dx$, thus we get

$$\int \frac{1}{x\log(x)^5}dx = \int \frac{1}{u^5}du = -\frac{1}{4u^4} + c = -\frac{1}{4\log(x)^4} + c,$$

   where c is an arbitrary constant.

3. Let's call their approval rating $A(t)$, so we have that $A(1) = 80$. We know that $A'(t) = \frac{-20}{t}$. Thus

$$A(t) = \int \frac{-20}{t}dt = -20\log(t) + c$$

   and thus to find c we plug in $A(1) = 80$, giving

$$\begin{aligned} -20\log(1) + c &= 80 \\ c &= 80 \end{aligned}$$

   Thus $A(t) = 80 - 20\log(t)$.

   At the end of the year, i.e. $t = 12$, we get $A(12) = 80 - 20\log(12) \approx 58.42$.

Deeper Thinking

1. The product rule says
$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).$$
Integrating this expression gives us
$$f(x)g(x) = \int f'(x)g(x)dx + \int f(x)g'(x)dx.$$
The way this becomes useful is we can arrange this as
$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx.$$
So if we have a product of functions in an integral, we can differentiate one and antidifferentiate the other to use the above formula. This often simplifies matters and is called *Integration By Parts*.

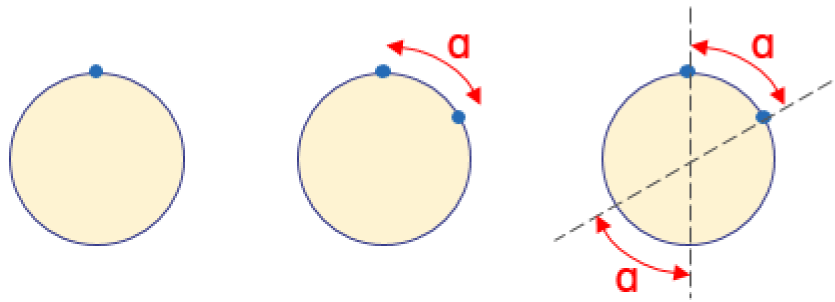
2. For this integral we consider $f(x) = x$ and $g(x) = \log(x)$. Then we have $f'(x) = 1$ and $g'(x) = \frac{1}{x}$. Thus Integration By Parts from Problem 1 gives us
$$\begin{aligned}\int \log(x)dx &= \int f'(x)g(x)dx \\ &= f(x)g(x) - \int f(x)g'(x)dx \\ &= x\log(x) - \int x\frac{1}{x}dx \\ &= x\log(x) - \int 1dx \\ &= x\log(x) - x + c,\end{aligned}$$
where c is an arbitrary constant.

3. To compute the probability, we are going to "add up" all the ways that you can make a triangle containing the center and divide by all possible triangles. To simplify calculations we will invoke various symmetries to avoid overcounting things.

   First of all, after the first point is placed, we can rotate the circle to fix it at the top. So we'll assume that one point is fixed at the top.

   The second point is some angle a clockwise from the top. We can assume it is on the right half of the triangle, so $a \in [0, \pi]$.

   Where can the third point lie to make sure that the resulting triangle contains the centre of the circle? If you join those two with a straight line, we see that the third point has to lie on an arc on the opposite side of the circle as pictured.



Thus if we fix the second point at an angle a, the allowable range of angles is also a. However, the third point could be anywhere around the circle, i.e. in a range of 2π. This means we need to "add" and divide these counts over all possible values of a, giving

$$
\begin{aligned}
\text{Probability of triangle containing the centre} &= \frac{\text{amount of triangles that contain the centre}}{\text{amount of all possible triangles}} \\
&= \frac{\displaystyle\int_0^\pi a da}{\displaystyle\int_0^\pi 2\pi da} \\
&= \frac{\frac{1}{2}a^2\Big|_0^\pi}{2\pi a|_0^\pi} \\
&= \frac{\frac{1}{2}\pi^2}{2\pi^2} \\
&= \frac{1}{4}
\end{aligned}
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