Problem Set 9 - Solutions

2019 Math Boot Camp for the Political and Social Sciences

Some practice

1. We have
$$\frac{\partial f}{\partial x} = 3x^2(y+1) + e^y$$
$$\frac{\partial f}{\partial y} = x^3 + xe^y$$

2. (a) $\nabla f = \langle 1, 1 \rangle$
   (b) $\nabla f = \langle \cos(x)\sin(y), \sin(x)\cos(y) \rangle$
   (c) $\nabla f = \langle e^{x+y^2}, 2ye^{x+y^2} \rangle$
   (d) $\nabla f = \langle 2xy^2 + y - 1, 2x^2y + x - 1 \rangle$
   (e) $\nabla f = \left\langle \frac{1}{x}\log(x+y) + \log(x)\frac{1}{x+y}, \log(x)\frac{1}{x+y} \right\rangle$

3. We have the constraint function $g(x, y) = x^2 + y^2 - 4$. Thus Lagrange Multipliers give the condition
$$\nabla f = \lambda \nabla g$$
$$\langle 2xy, x^2 + 2 \rangle = \langle 2\lambda x, 2\lambda y \rangle$$
So we have the three equations
$$2xy = 2\lambda x \qquad x^2 + 2 = 2\lambda y \qquad x^2 + y^2 = 4$$
The first equation implies either $x = 0$ or $\lambda = y$. If $x = 0$ then the third equation gives $y = \pm 2$, so we have extreme values at the points $(0, 2)$ and $(0, -2)$.
If $y = \lambda$, the the second equation rearranges to $x^2 = 2y^2 - 2$, and we plug that into the third equation to get
$$2y^2 - 2 + y^2 = 4$$
$$3y^2 = 6$$
$$y = \pm\sqrt{2}$$
Plugging these into the last give $x = \pm\sqrt{2}$. Thus our extreme values occur at the points
$$(0, 2) \qquad (0, -2) \qquad (\sqrt{2}, \sqrt{2}) \qquad (-\sqrt{2}, \sqrt{2}) \qquad (\sqrt{2}, -\sqrt{2}) \qquad (-\sqrt{2}, -\sqrt{2})$$
These give the following extreme values for $f(x, y)$
$$f(0, 2) = 4 \qquad f(\sqrt{2}, \sqrt{2}) = f(-\sqrt{2}, \sqrt{2}) = 4\sqrt{2}$$
$$f(0, -2) = -4 \qquad f(\sqrt{2}, -\sqrt{2}) = f(-\sqrt{2}, -\sqrt{2}) = -4\sqrt{2}.$$

4. We are trying to maximise $f(x,y) = xy$ subject to the constraint function $g(x,y) = x+y-20$. Thus Lagrange Multipliers give the condition

$$\begin{aligned}\nabla f &= \lambda \nabla g \\ \langle y, x\rangle &= \langle \lambda, \lambda\rangle\end{aligned}$$

So we have these three equations

$$y = \lambda \qquad\qquad x = \lambda \qquad\qquad x + y = 20$$

The first two equations imply $x = y$, and plugging into the third gives $x = 10$ and $y = 10$, thus the largest value of the product is $10 \times 10 = 100$.

5. The total time taken is

$$F(x_1, x_2) = f_1(x_1) + f_2(x_2) = x_1 + 5x_1^2 + x_2 + 3x_2^2$$

and we want to minimize this subject to the constraint function $g(x_1, x_2) = x_1 + x_2 - 1000$. Thus Lagrange Multipliers give the condition

$$\begin{aligned}\nabla f &= \lambda \nabla g \\ \langle 1 + 10x_1, 1 + 6x_2\rangle &= \langle \lambda, \lambda\rangle\end{aligned}$$

So we have these three equations

$$1 + 10x_1 = \lambda \qquad\qquad 1 + 6x_2 = \lambda \qquad\qquad x_1 + x_2 = 1000$$

The first two equations give

$$\begin{aligned}1 + 10x_1 &= 1 + 6x_2 \\ x_1 &= \frac{3}{5}x_2\end{aligned}$$

Then plugging into the third gives

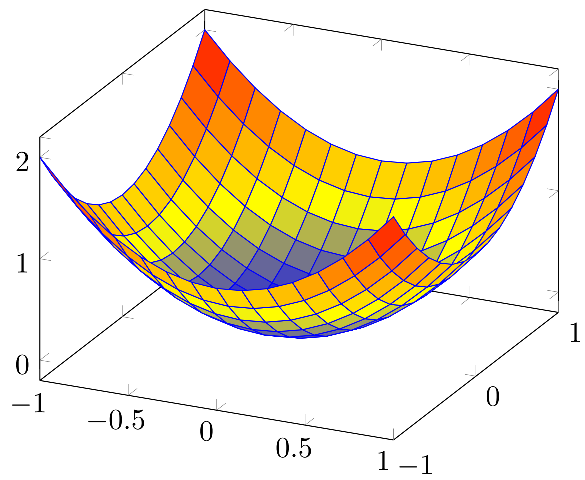
$$\begin{aligned}\frac{3}{5}x_2 + x_2 &= 1000 \\ \frac{8}{5}x_2 &= 1000 \\ x_2 &= \frac{5000}{8} = 625\end{aligned}$$

and thus $x_1 = \frac{3}{5} \times 625 = 375$. Thus the minimum total amount of time spent is

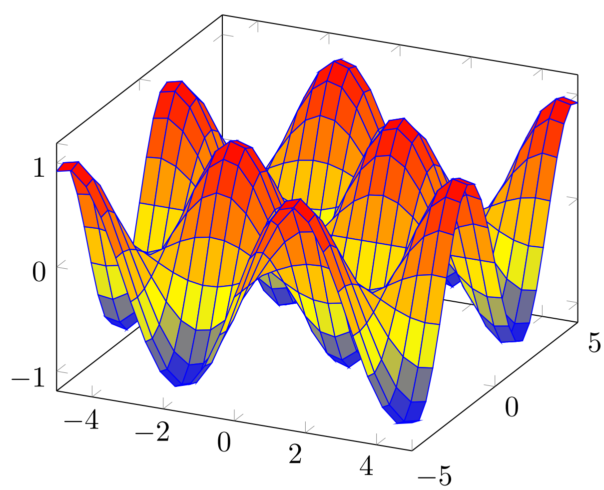
$$F(375, 625) = f_1(375) + f_2(625) = 1,876,000 \text{ seconds.}$$

Which is ≈ 21.71 days.

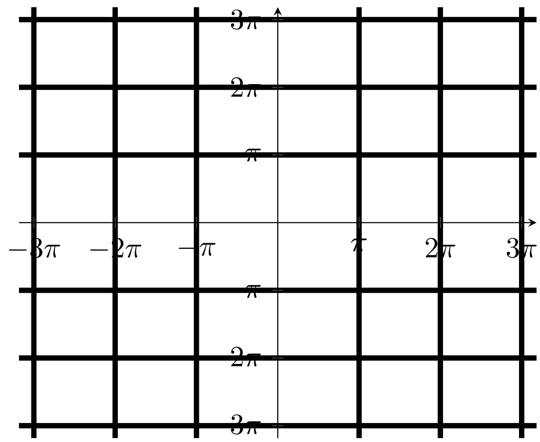
Deeper Thinking

1. 

2. The graph of $f(x) = \sin(x)\sin(y)$ looks like



If we look at the set $\sin(x)\sin(y) = 0$, we see either $x = \ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$ or $y = \ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$. Thus the set of points $f(x, y) = 0$ is


3π
2π
π
-3π -2π $-\pi$ π 2π 3π
π
2π
3π