MA 122 Quiz 1– Monday, July 11, 2011

In order to receive full credit, please show all work supporting your answer. All derivatives and integrals must be computed by hand, without the aid of a calculator. Good luck!

1. Find a function $y = f(x)$ that satisfies both conditions:

$$\frac{dy}{dx} = 9x^2 e^{x^3} \quad f(0) = 2.$$

(Hint: Use substitution.)

Solution.

$$\int \frac{dy}{dx} dx = \int 9x^2 e^{x^3} dx$$

To integrate, we use substitution: Let $u = x^3$ so that $du = 3x^2 dx$ and $x^2 dx = \frac{1}{3} du$. Then the integral above becomes

$$\begin{aligned} \int 9x^2 e^{x^3} dx &= \int 3e^u du \\ &= 3e^u + C \\ &= 3e^{x^3} + C. \end{aligned}$$

So $f(x) = 3e^{x^3} + C$ and we use $f(0) = 2$ to determine C:

$$\begin{aligned} f(0) = 2 &\Rightarrow 3e^0 + C = 2 \\ &\Rightarrow 3 + C = 2 \\ &\Rightarrow C = -1 \\ &\Rightarrow f(x) = 3e^{x^3} - 1 \end{aligned}$$

2. (a) Find the producers' surplus at a price level $\bar{p} = \$5.545$ for the price-supply equation

$$p = S(x) = (x + 1)\ln(x + 1).$$

Round your answer to three decimal places.

Solution to part (a). In class, I told you to use $\bar{x} = 3$. Then the equation for producers' surplus is

$$\begin{aligned}\int_0^{\bar{x}} [\bar{p} - S(x)]\, dx &= \int_0^3 [5.545 - (x+1)\ln(x+1)]\, dx \\ &= 5.545x\Big|_0^3 - \int_0^3 (x+1)\ln(x+1)dx \\ &= 16.635 - \int_0^3 (x+1)\ln(x+1)dx.\end{aligned}$$

There are a number of ways to compute the second integral above involving the natural logarithm. I'll list two here.

Method 1. Start with substitution: Let $u = (1+x)$ so that $du = dx$. Then

$$\int_0^3 (x+1)\ln(x+1)dx = \int_1^4 u \ln u du$$

and now we can readily use integration by parts: Let $\tilde{u} = \ln u$ and $dv = u$. Then $d\tilde{u} = \frac{1}{u}du$ and $v = \frac{u^2}{2}$, which we plug into the integration by parts formula:

$$\begin{aligned}\int_1^4 u \ln u du &= \ln u \frac{u^2}{2}\Big|_1^4 - \int_1^4 \frac{u^2}{2}\frac{1}{u}du \\ &= (8\ln 4 - 0) - \frac{1}{2}\int_1^4 udu \\ &= 8\ln 4 - \frac{1}{2}\frac{u^2}{2}\Big|_1^4 \\ &= 8\ln 4 - (4 - \frac{1}{4}) \\ &= 8\ln 4 - 3.75\end{aligned}$$

Method 2. Start with integration by parts: Let $u = \ln(1+x)$ and

$dv = (x+1)dx$ so that $du = \frac{1}{1+x}dx$ and $v = \frac{x^2}{2} + x$. Plugging in, we get

$$\begin{aligned}
\int_0^3 (x+1)\ln(x+1)dx &= \ln(1+x)\left(\frac{x^2}{2}+x\right)\Big|_0^3 - \int_0^3 \left(\frac{x^2}{2}+x\right)\frac{1}{1+x}dx \\
&= \ln 4 * (\frac{9}{2}+3) - \int_0^3 \frac{\frac{x^2}{2}+x}{1+x}dx \\
&= 7.5\ln 4 - \int_0^3 \frac{\frac{x^2}{2}+x}{1+x}dx.
\end{aligned}$$

Now we use substitution to compute the remaining integral: Set $u = x+1$ so that $du = dx$ and $x = u-1$. Then

$$\begin{aligned}
\int_0^3 \frac{\frac{x^2}{2}+x}{1+x}dx &= \int_1^4 \frac{\frac{(u-1)^2}{2}+(u-1)}{u}du \\
&= \int_1^4 \frac{\frac{u^2}{2}-u+\frac{1}{2}+u-1}{u}du \\
&= \int_1^4 \left(\frac{u}{2}-\frac{\frac{1}{2}}{u}\right)du \\
&= \left(\frac{u^2}{4}-\frac{1}{2}\ln u\right)\Big|_1^4 \\
&= \left(4-\frac{1}{2}\ln 4\right) - \left(\frac{1}{4}-0\right) \\
&= 3.75 - 0.5\ln 4
\end{aligned}$$

Now we put it all together to get

$$\begin{aligned}
\int_0^3 (x+1)\ln(x+1)dx &= 7.5\ln 4 - (3.75 - 0.5\ln 4) \\
&= 8\ln 4 - 3.75
\end{aligned}$$

which is the same answer we found using Method 1.

There is at least one more way to do this problem, and you got full credit if you used another method and reached the correct answer. Finally, we put the pieces together to get the producers' surplus:

$$
\begin{aligned}
PS &= \int_0^3 [5.545 - (x+1)\ln(x+1)]\,dx \\
&= 16.635 - \int_0^3 (x+1)\ln(x+1)dx \\
&= 16.635 - (8\ln 4 - 3.75) \\
&= 20.385 - 8\ln 4 \\
&\approx 9.295
\end{aligned}
$$

The **producers' surplus at price level** $\bar{p} = \$5.545$ **is therefore** \$9.295.

(b) Draw a graph which illustrates your answer to part (a). Be sure to include any relevant labels (e.g. axes, important points).