MA 122– Exam 2

August 1, 2011

In order to receive full credit, please show all work– You may use a calculator, but all derivatives and integrals must be computed by hand. Good luck!

Problem 1 Compute the following derivatives and integrals. When evaluating definite integrals, give exact answers with values taken from the unit circle (i.e. no decimal approximations). Be sure to show your work.

(a) $\frac{d}{dx}\left(\cos(x^3)\right)$

Solution Use the chain rule:

$$\begin{aligned} &= (-\sin(x^3))(3x^2) \\ &= -3x^2\sin(x^3) \end{aligned}$$

(b) $\frac{d}{dx}\left(\cos^3(x)\right)$

Solution

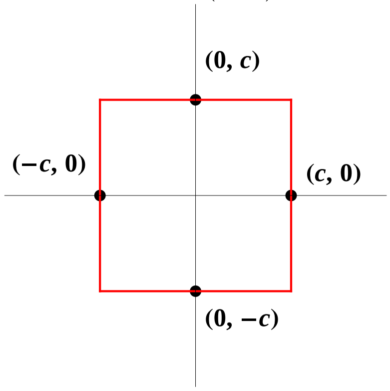
$$\begin{aligned} &= (3\cos^2(x))(-\sin(x)) \\ &= -3\sin(x)\cos^2(x) \end{aligned}$$

(c) $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{\sin\theta}{\cos^3\theta}d\theta$

Solution Use substitution: let $u = \cos\theta$ so that $-du = \sin\theta d\theta$ and

$$\begin{aligned} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{\sin\theta}{\cos^3\theta}d\theta &= \int_{\sqrt{2}/2}^{-\sqrt{2}/2} -\frac{1}{u^3}du \\ &= \frac{1}{2}u^{-2}\Big|_{\sqrt{2}/2}^{-\sqrt{2}/2} \\ &= 0 \end{aligned}$$

Problem 2 Find the dimensions of the square S centered at the origin for which the average value of $f(x, y) = x^2y^2$ over S is equal to 100.



Solution From the average value formula, we know that the average value of f over the square S is equal to

$$\begin{aligned}
\frac{1}{\text{Area of } S}\int\int_S f(x, y)dA &= \frac{1}{(2c)(2c)}\int_{-c}^{c}\int_{-c}^{c} x^2y^2dxdy \\
&= \frac{1}{4c^2}\int_{-c}^{c} y^2\left(\frac{c^3}{3} - \frac{-c^3}{3}\right)dy \\
&= \frac{c}{6}\left(\frac{c^3}{3} - \frac{-c^3}{3}\right) \\
&= \frac{c^4}{9}.
\end{aligned}$$

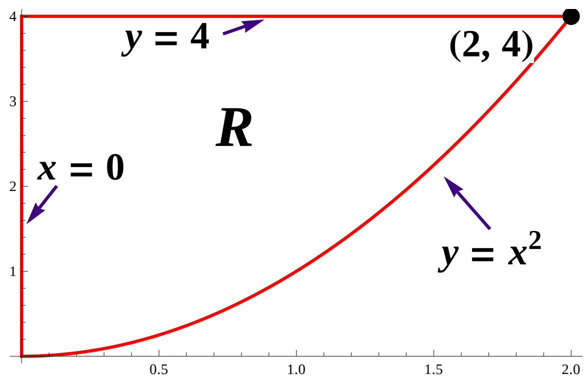
We want to find c so that the average value is equal to 100, i.e.

$$\frac{c^4}{9} = 100$$

and so $c = \sqrt{30} \approx 5.477$. The dimensions of the square are $2c$ by $2c$, which is approximately 10.955 by 10.955.

Problem 3 Consider $\int_0^2 \int_{x^2}^4 \frac{4x}{1+y^2} dy dx$. (a) Sketch a graph of the region of integration, and then describe it as *both* a regular x region and a regular y region. (b) Evaluate the double integral in any way that you can (without a calculator).

Solution



Regular x region: $\{(x, y)|x^2 \leq y \leq 4, \;\; 0 \leq x \leq 2\}$

Regular y region: $\{(x, y)|0 \leq x \leq \sqrt{y}, \;\; 0 \leq y \leq 4\}$

In order to evaluate the integral, we switch the order of integration using the limits from the regular y region we found above.

$$\begin{aligned}\int_0^2 \int_{x^2}^4 \frac{4x}{1+y^2} dy dx &= \int_0^4 \int_0^{\sqrt{y}} \frac{4x}{1+y^2} dx dy \\ &= \int_0^4 \frac{4}{1+y^2} \left[\int_0^{\sqrt{y}} x dx\right] dy \\ &= \int_0^4 \frac{2}{1+y^2} \left[x^2\Big|_0^{\sqrt{y}}\right] dy \\ &= \int_0^4 \frac{2y}{1+y^2} dy\end{aligned}$$

Now we use substitution on the remaining integral:

$$\begin{aligned}\int_0^4 \frac{2y}{1+y^2} &= \int_1^{17} \frac{1}{u} du \\ &= \ln|u|\Big|_1^{17} \\ &= \ln(17).\end{aligned}$$

Problem 4 Revenues from boat shoe sales in a particular store are given approximately by

$$R(t) = 3 + 2\cos\left(\frac{\pi t}{6}\right), \quad 0 \le t \le 24$$

where $R(t)$ is the revenue in thousands of dollars for a month of sales t months after June 1.

(a) Find the exact value $R(2)$ and interpret the result in the context of this problem.

Solution $R(2) = 4$. This means that the revenue taken in by the store from boat shoe sales was approximately \$4000 during the month of August (i.e. two months after June 1).

(b) Find all local maxima and minima of R for $0 < t < 24$.

Solution We first find the critical points of R by setting $R'(t) = 0$:

$$\begin{aligned} R'(t) &= -\frac{2\pi}{6}\sin\left(\frac{\pi t}{6}\right) \\ R'(t) &= 0 \\ \Rightarrow \sin\left(\frac{\pi t}{6}\right) &= 0 \\ \Rightarrow \frac{\pi t}{6} &= k\pi, \ k \in \mathbb{Z}^+ \\ \Rightarrow t &= 6k. \end{aligned}$$

Since we are restricting to $0 < t < 24$, we see that $0 < 6k < 24 \Rightarrow 0 < k < 4$. In other words, $k = 1$, 2, or 3 so that there are exactly three critical points: $t = 6$, $t = 12$ and $t = 18$.

To determine which critical points are maxima and which are minima, we compute $R''(t)$ and evaluate it at the three points.

$$\begin{aligned} R''(t) &= -\frac{2\pi^2}{6^2}\cos\left(\frac{\pi t}{6}\right) \\ &= -\frac{\pi^2}{18}\cos\left(\frac{\pi t}{6}\right) \end{aligned}$$

We find that

$$\begin{aligned} R''(6) &= -\frac{\pi^2}{18}\cos\pi \\ &= \frac{\pi^2}{18} \\ &> 0 \end{aligned}$$

and so $t = 6$, $R = 1$ is a *local minimum*. Next,

$$\begin{aligned} R''(12) &= -\frac{\pi^2}{18}\cos(2\pi) \\ &= -\frac{\pi^2}{18} \\ &< 0 \end{aligned}$$

so that $t = 12$, $R = 5$ is a *local maximum.* Finally,

$$\begin{aligned} R''(18) &= -\frac{\pi^2}{18}\cos(3\pi) \\ &= \frac{\pi^2}{18} \\ &> 0 \end{aligned}$$

and so $t = 18$, $R = 1$ is another *local minimum.*

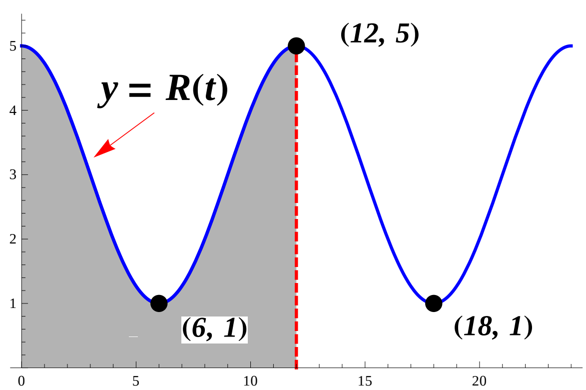
(c) How much revenue did the store take in during the first year?

Solution

$$\begin{aligned} \int_0^{12} \left(3 + 2\cos\left(\frac{\pi t}{6}\right)\right) dt &= \left(3t + \frac{12}{\pi}\sin\left(\frac{\pi t}{6}\right)\right)\Big|_0^{12} \\ &= \left(36 + \frac{12}{\pi}\sin(2\pi)\right) - \left(0 + \frac{12}{\pi}\sin(0)\right) \\ &= 36 \end{aligned}$$

This tells us that the store took in $\$36,000$ in revenue for boat shoes during the first year.

(d) Use part (b) to help you sketch a graph of $R(t)$, and label all points found in part (b). Indicate the quantity found in part (c) on your graph. Be sure to include all important labels.



The shaded region in the graph has area equal to 36, the number found in part (c) above.

Problem 5 (Extra credit to be added to Quiz 2 or Exam 2– whichever produces the better grade) Give an example of a real-world situation (*other than revenue*) that could be modeled by a periodic function, and explain why. Estimate the period and explain your choice. Sketch and label a corresponding graph.

Answer will vary widely. See me if you would like to talk about this problem.