MA 122– Quiz 2

July 24, 2011

In order to receive full credit, please show all work– You may use a calculator, but all derivatives and integrals must be computed by hand. Good luck!

Problem 1 Find the average value of the function $f(x,y) = (x+y)^2$ over the region

$$R = \{(x,y) | 1 \leq x \leq 5,\ -1 \leq y \leq 1\}.$$

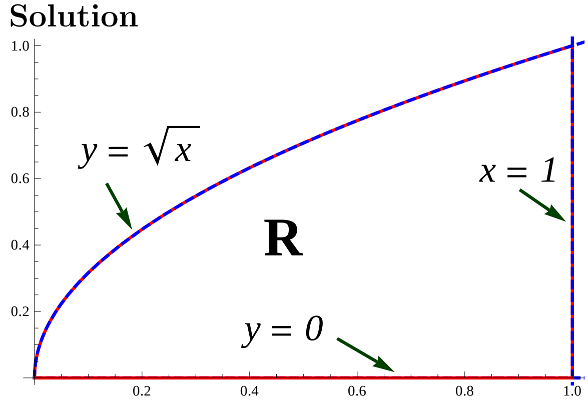
Solution The average value of the function f over the given rectangle is

$$\begin{aligned}
\frac{1}{(5-1)(1-(-1))} \int_1^5 \int_{-1}^1 (x+y)^2 dy dx &= \frac{1}{8} \int_1^5 \left[\frac{(x+y)^3}{3}\right] \Big|_{y=-1}^{1} dx \\
&= \frac{1}{8} \int_1^5 \left(\frac{(x+1)^3}{3} - \frac{(x-1)^3}{3}\right) dx \\
&= \frac{1}{8} \left[\frac{(x+1)^4}{12} - \frac{(x-1)^4}{12}\right] \Big|_1^5 \\
&= \frac{1}{8} \left[\left(\frac{6^4}{12} - \frac{4^4}{12}\right) - \left(\frac{2^4}{12} - 0\right)\right] \\
&= \frac{32}{3} \approx 10.67
\end{aligned}$$

Problem 2 Consider $\int_0^1 \int_{y^2}^1 4ye^{x^2} dx dy$.

(a) Graph the regular region R which is represented by the limits of integration, and describe it as both a regular x region *and* a regular y region.

Solution



Regular y region: $R = \{(x,y) | y^2 \leq x \leq 1,\ 0 \leq y \leq 1\}$

Regular x region: $R = \{(x,y) | 0 \leq y \leq \sqrt{x},\ 0 \leq x \leq 1\}$

(b) Evaluate the integral.

Solution Since $\int e^{x^2} dx$ is very hard to compute and since we know both substitution and integration by parts won't work, we have to *switch the order of integration*. To do this, we can use the regular x region we found in part (a):

$$
\begin{aligned}
\int_0^1 \int_{y^2}^1 4ye^{x^2}\,dxdy &= \int_0^1 \int_0^{\sqrt{x}} 4ye^{x^2}\,dydx \\
&= \int_0^1 4e^{x^2} \left[\int_0^{\sqrt{x}} ydy \right] dx \\
&= \int_0^1 4e^{x^2} \left[\frac{y^2}{2} \Big|_{y=0}^{\sqrt{x}} \right] dx \\
&= \int_0^1 2xe^{x^2}\,dx.
\end{aligned}
$$

Now we are set up to use the method of substitution on the remaining integral. Let $u = x^2$ so that $du = 2xdx$. Then

$$
\begin{aligned}
\int_0^1 2xe^{x^2}\,dx &= \int_0^1 e^u du \\
&= e^u \Big|_{u=0}^{1} \\
&= e - 1.
\end{aligned}
$$

(c) Explain what the number found in part (b) represents. Justify your answer (using, for example, theorems from class or your own logic).

Solution It was stated as a theorem in class that for a nonnegative function $f(x,y)$ over a regular region, the double integral of f over that region represents the *volume of the solid formed by graphing f over the region*. Since $f(x,y) = 4ye^{x^2} \geq 0$ over the region R described above, it follows that the number found in part (b) represents the volume of the solid formed by graphing f over R. For illustrative purposes, I've included a graph below (this was not expected of you on the quiz).

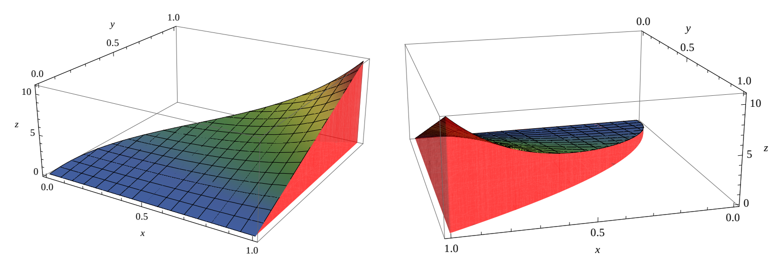


Figure 1: Back and front view of the solid (red) which has volume $e - 1 \approx 1.17$. The "bottom" of the solid is the region R and the "top" of the solid is the graph of $f(x, y)$.