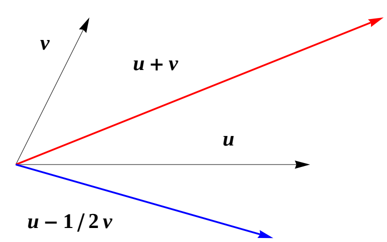
Name:

Exam 1

MA 225 A1, 6/1/12

In order to receive full credit, you must show all work. No calculators are allowed, but you may use a 3×5 note card. Good luck!

1. Given the vectors $\vec{u}$ and $\vec{v}$ below, draw and label the vectors $\vec{u}+\vec{v}$ and $\vec{u}-\frac{1}{2}\vec{v}$.



2. (a) Find *all* vectors $\vec{u} = \langle 1, 1, a\rangle$ that are orthogonal to $\vec{v} = \langle 2, 0, 1\rangle$.

Solution. $\vec{u}$ and $\vec{v}$ are orthogonal provided $\vec{u} \cdot \vec{v} = 0$. Therefore, we need $2 + 0 + a = 0 \Rightarrow a = -2$. Thus $\vec{u} = \langle 1, 1, -2\rangle$.

(b) Use your solution from part (a) to compute *two* vectors which are orthogonal to both $\vec{u}$ and $\vec{v}$.

Solution. The cross product of two nonzero vectors yields a third vector orthogonal to both vectors. Moreover, any nonzero scalar multiple of this third vector will be orthogonal to both $\vec{u}$ and $\vec{v}$. We compute

$$\begin{aligned}\vec{u}\times\vec{v} &= \langle 1,1,-2\rangle \times \langle 2,0,1\rangle \\ &= \langle 1-0, -4-1, 0-2\rangle \\ &= \langle 1,-5,-2\rangle\end{aligned}$$

Thus two vectors that are orthogonal to $\vec{u}$ and $\vec{v}$ are $\langle 1,-5,-2\rangle$ and $\langle -1,5,2\rangle$, but of course these are not the only two such vectors.

3. Let $\vec{v} = -8\vec{i} - 6\vec{j}$.

(a) Find a unit vector parallel to $\vec{v}$.

Solution. $|\vec{v}| = \sqrt{64+36} = 10$ so that one possible unit vector parallel to $\vec{v}$ is

$$\begin{aligned}\frac{\vec{v}}{|\vec{v}|} &= \frac{\langle -8,-6\rangle}{10} \\ &= -\frac{4}{5}\vec{i} - \frac{3}{5}\vec{j}.\end{aligned}$$

(b) Suppose $\vec{u}$ has length 3 and is parallel to $\vec{v}$. Give an expression for $\vec{u}$.

Solution. Since the vector in part (a) has length one, three times that vector will give us a vector of length 3 that is parallel to $\vec{v}$. So one possible choice of $\vec{u}$ is

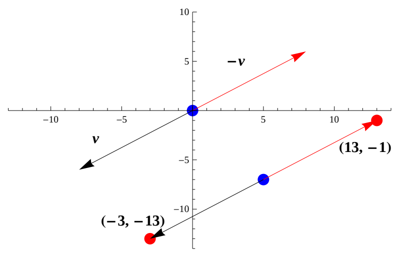
$$\vec{u} = -\frac{12}{5}\vec{i} - \frac{9}{5}\vec{j}.$$

(c) Suppose $\vec{v}$ is parallel to $\vec{AB}$ (the vector with tail at A and head at B) and both vectors have the same magnitude. Find the point B if A is $(5,-7)$. Sketch both vectors in the plane.

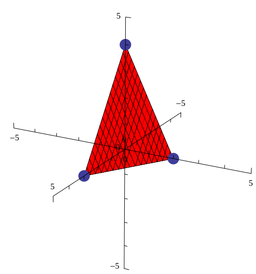
Solution. There are two possible choices for the point B: the first is assuming that $\vec{AB}$ points in the same direction as $\vec{v}$ and the second is found by assuming that $\vec{AB}$ points in the direction of $-\vec{v}$. Let's proceed with the first choice. Let $B = (x,y)$. Then $\vec{AB} = \langle x-5, y+7\rangle$, and we would like to choose x and y so that

$$\begin{aligned}\langle x-5, y+7\rangle &= \langle -8,-6\rangle\\ &\Rightarrow x=-3,\quad y=-13.\end{aligned}$$

So one choice of the point B is $(-3,-13)$ (the other possible choice is $(13,-1)$).



4. (a) Draw the triangle in $\mathbb{R}^3$ with vertices at $(3,0,0)$, $(0,2,0)$, and $(0,0,4)$.



(b) Explain how one could use the cross product to find the area of this triangle.

Solution. Label the vertices of the triangle $A=(3,0,0)$, $B=(0,2,0)$, $C=(0,0,4)$. Then we can find the area of the triangle by taking half the magnitude of the cross product of any two vectors formed by connecting any two of the vertices A, B and C, provided that the vectors have the same tail (i.e. they “start” at the same point).

More explicitly, we could for example compute the area of the triangle to be $\frac{1}{2}|\vec{AB} \times \vec{AC}|$.

(c) Carry out your explanation from part (b) to calculate the area of the triangle.

Solution. $\vec{AB} = \langle -3, 2, 0\rangle$, $\vec{AC} = \langle -3, 0, 4\rangle$, so $\vec{AB} \times \vec{AC} = \langle 8 - 0, 0 - -12, 0 - -6\rangle = \langle 8, 12, 6\rangle$. The magnitude of this vector is $\sqrt{64 + 144 + 36} = \sqrt{244}$. Thus the area of the triangle is $\frac{1}{2}\sqrt{244}$.

5. (a) Compute the indefinite integral $\int(\sqrt{t}\,\vec{i} + t^{-3}\vec{j} - 2t^2\vec{k})dt$

Solution.

$$\frac{2}{3}t^{3/2}\vec{i} - \frac{1}{2t^2}\vec{j} - \frac{2}{3}t^3\vec{k} + \vec{C}$$

where $\vec{C}$ is a constant vector.

(b) Let $\vec{u}(t) = 2t^3\vec{i} + (t^2 - 1)\vec{j} - 8\vec{k}$. Compute the derivative of $(t^{12} + 3t)\vec{u}(t)$.

Solution. We use the product rule to compute

$$\begin{aligned}\frac{d}{dt}\left[(t^{12} + 3t)\vec{u}(t)\right] &= (12t^{11} + 3)\vec{u}(t) + (t^{12} + 3t)\vec{u}'(t) \\ &= (12t^{11} + 3)(2t^3\vec{i} + (t^2 - 1)\vec{j} - 8\vec{k}) + (t^{12} + 3t)(6t^2\vec{i} + 2t\vec{j}).\end{aligned}$$

If you reached this point then you received full credit– no need to do all of the simplification.

6. (a) Find an equation for a line that passes through the point $(0, 1, 1)$ and is parallel to $\langle 2, -2, 2\rangle$.

Solution In vector-valued function notation: $\vec{r}(t) = \langle 2t, 1 - 2t, 1 + 2t\rangle$, $-\infty < t < \infty$.

(b) State the formula for the arclength of a curve $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ from $t = a$ to $t = b$.

Solution. Arclength $= \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}dt$

(c) Find the length of the segment of the line you found in part (a) from $t = -2$ to $t = 2$.

Solution. Using the formula from part (b), we have

$$\begin{aligned}\int_{-2}^{2} \sqrt{2^2 + (-2)^2 + 2^2}dt &= \int_{-2}^{2} \sqrt{12}dt \\ &= 4\sqrt{12} \\ &= 8\sqrt{3}.\end{aligned}$$