Name: KEY

Exam 2, MA 225 A1, 6/14/12

In order to receive full credit, you must show all work. No calculators are allowed, but you may use a 3×5 note card. Good luck!

1. Find the equation of a plane containing the points $(-2,3,1)$, $(1,1,0)$ and $(-1,0,1)$.

ANSWER. In order to find the equation of the plane, we need a point in the plane and a vector normal to it. We are given three possible points, and we can create a normal vector by taking the cross product of two vectors formed by choosing one point to be the "tail" and the other two to be the "heads" of the vectors.

For instance, let $A = (-2,3,1)$, $B = (1,1,0)$ and $C = (-1,0,1)$. Then $\overrightarrow{AB} = \langle 3,-2,-1\rangle$ and $\overrightarrow{AC} = \langle 1,-3,0\rangle$. A normal vector to the plane is given by

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \langle -3,-1,-7\rangle$$

Thus the equation for the plane can be written

$$-3(x+2) - (y-3) - 7(z-1) = 0$$

or

$$3x + y + 7z = 4.$$

2. (a) Suppose $V = V(r(t), h(t))$. Find an expression for $\frac{dV}{dt}$.

ANSWER. Use the chain rule: $\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt}$.

(b) If $V(r,h) = \pi r^2 h$ and $r = e^t$, $h = e^{-2t}$, *use the formula you found in part (a)* to calculate $\frac{dV}{dt}$.

ANSWER. $\frac{\partial V}{\partial r} = 2\pi r h$, $\frac{\partial V}{\partial h} = \pi r^2$, $\frac{dr}{dt} = e^t$, and $\frac{dh}{dt} = -2e^{-2t}$. Thus

$$\begin{aligned}\frac{dV}{dt} &= \left(2\pi e^t e^{-2t}\right)\left(e^t\right) + \left(\pi e^{2t}\right)\left(-2e^{-2t}\right)\\ &= 2\pi e^0 - 2\pi e^0\\ &= 0.\end{aligned}$$

3. Let $f(x,y) = x^2 + 2xy - y^3$.

(a) Compute all first and second partial derivatives of f: f_x, f_y, f_{xx}, f_{yy} and f_{xy}.

ANSWER.

$$\begin{aligned}f_x &= 2x + 2y\\ f_y &= 2x - 3y^2\\ f_{xx} &= 2\\ f_{yy} &= -6y\\ f_{xy} &= 2\end{aligned}$$

(b) Find all critical points of f.

ANSWER. We must solve $f_x = 0$ and $f_y = 0$ simultaneously.

$$\begin{aligned} f_x &= 2x + 2y = 0 \ \Rightarrow y = -x \\ \Rightarrow f_y &= 2x - 3(-x)^2 \\ &= x(2 - 3x) \\ &= 0 \ \Rightarrow \ x = 0 \ \text{ or } \ x = \frac{2}{3} \end{aligned}$$

Thus there are two critical points: $(0, 0)$ and $\left(\frac{2}{3}, -\frac{2}{3}\right)$.

(c) Classify each critical point as a local maximum, local minimum, or saddle point.

ANSWER.
We must first compute the discriminant and evaluate it at each point: $D(x, y) = 2(-6y) - (2)^2 = -12y - 4$

Since $D(0, 0) = -4 < 0$, according to the second derivative test the critical point $(0, 0)$ is a **saddle point**.

Since $D\left(\frac{2}{3}, -\frac{2}{3}\right) = 4 > 0$, and $f_{xx} = 2 > 0$ we see that $\left(\frac{2}{3}, -\frac{2}{3}\right)$ must be a **local minimum**.

4. Let $f(x, y) = -\sqrt{4 - x^2 - y^2}$
(a) Compute $\nabla f(x, y)$ and evaluate $\nabla f(-1, 1)$.

ANSWER. $\nabla f(x, y) = \langle \frac{x}{\sqrt{4-x^2-y^2}}, \frac{y}{\sqrt{4-x^2-y^2}} \rangle$, so $\nabla f(-1, 1) = \langle \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$.

(b) Find the unit vectors that give the direction of steepest ascent and steepest descent at $(-1, 1)$.

ANSWER. We know that the gradient vector points in the direction of steepest ascent and its opposite points in the direction of steepest descent at a given point. Since $\nabla f(-1, 1)$ is a unit vector, we have the direction of steepest ascent is $\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$ while the direction of steepest descent is $\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle$.

(c) Find a unit vector that points in a direction of no change.

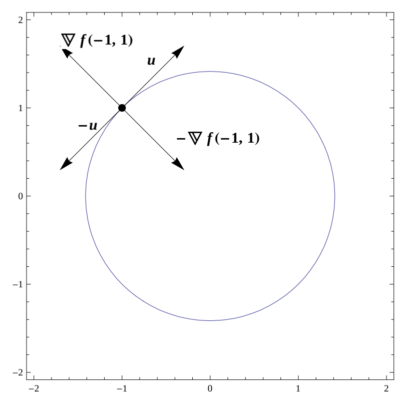
ANSWER. We know that such a vector must be orthogonal to the gradient vector, i.e. $\vec{u} = \langle u_1, u_2 \rangle$ with $\nabla f(-1, 1) \cdot \vec{u} = 0$. Therefore,

$$\begin{aligned} -\frac{1}{\sqrt{2}} u_1 + \frac{1}{\sqrt{2}} u_2 = 0 \\ \Rightarrow u_1 = u_2 \end{aligned}$$

and so we can take $\vec{u} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$. We could also take $-\vec{u}$.

(d) Sketch the level curve of f that contains the point $(x, y) = (-1, 1)$ and identify the corresponding value of z. On this curve sketch the unit vectors you identified in parts (b) and (c).

ANSWER. $f(-1,1) = -\sqrt{2}$, and so this is the value of z corresponding to the desired level curve. Setting $-\sqrt{2} = -\sqrt{4-x^2-y^2}$ will give the equation for the level curve in the xy-plane. Squaring both sides, we find $2 = 4 - x^2 - y^2$ which we rearrange as $x^2 + y^2 = 2$. Thus the level curve is a circle of radius $\sqrt{2}$:



(e) Identify the graph of the surface $z = f(x,y)$ in words, i.e. what is it called?

ANSWER. The graph is the bottom half of the sphere of radius 2 centered at the origin.

(f) Find the equation of the tangent plane to f at the point $(-1,1)$.

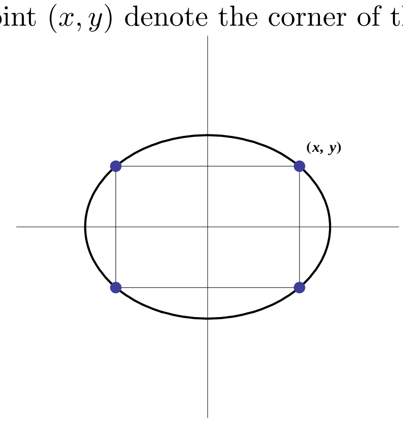
ANSWER. The tangent plane to $z = f(x,y)$ at a point $(a,b,f(a,b))$ is described by the equation

$$z = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b),$$

so the tangent plane at $(-1,1,-\sqrt{2})$ is

$$z = -\frac{1}{\sqrt{2}}(x+1) + \frac{1}{\sqrt{2}}(y-1) - \sqrt{2}.$$

5. Consider a rectangle inscribed in the ellipse $\frac{x^2}{4} + y^2 = 1$, as shown below where we let the point (x,y) denote the corner of the rectangle in the first quadrant.



(a) Write an equation for the perimeter of the rectangle in terms of the coordinates x and y.

ANSWER. $P(x,y) = 4x + 4y$

(b) Use the method of Lagrange multipliers to find the coordinates x and y that maximize the perimeter of such a rectangle.

ANSWER. We want to maximize $P(x, y)$ from part (a) subject to the constraint $g(x, y) = \frac{x^2}{4} + y^2 - 1 = 0$. Our Lagrange multiplier equations are:

$$\begin{aligned} 4 &= \lambda \frac{x}{2} \\ 4 &= \lambda 2y \\ \frac{x^2}{4} + y^2 - 1 &= 0. \end{aligned}$$

We have assumed $x, y > 0$. Thus we can solve the first two equations for λ and set them equal to one another:

$$\begin{aligned} \lambda &= \frac{8}{x} \\ \lambda &= \frac{2}{y} \\ \Rightarrow 4y &= x. \end{aligned}$$

Substituting this into the constraint equation we find

$$\begin{aligned} &\frac{(4y)^2}{4} + y^2 - 1 = 0 \\ \Rightarrow &5y^2 - 1 = 0 \\ \Rightarrow &y = \pm \frac{1}{\sqrt{5}}. \end{aligned}$$

However, only positive y makes sense for this problem. Thus the perimeter is maximized when $y = \frac{1}{\sqrt{5}}$ and $x = 4y = \frac{4}{\sqrt{5}}$.