Day 10: July 13th

- **Chapter: 2.2 The Geometry of Systems**
  - Homework:
    - Page 182 #1, 3, 9, 11-27 odd.
- **Chapter: 2.3 Analytical Methods**
  - Homework:
    - Page 167 #19.
    - Page 196 #1, 3, 5, 11, 15, 17.
  - Chapter 2: Review Problems:
    - Page 220 #1, 3, 5, 7, 13, 17, 19-24.

Predator-Prey Systems

- Assumptions
- System of equations:

$$\frac{dR}{dt} = k \cdot R - \alpha \cdot R \cdot F$$

$$\frac{dF}{dt} = -l \cdot F + \beta \cdot R \cdot F$$

Predator-Prey Systems

- Assume: $\alpha = \beta = k = l = 1,$

$$\frac{dR}{dt} = R \cdot \Big(1 - F\Big)$$

$$\frac{dF}{dt} = F \cdot \Big(-1 + R\Big)$$

can't get further analytically

- Equilibrium Points.

$$\Big(R(t), F(t)\Big) = (0, 0)$$

$$\Big(R(t), F(t)\Big) = (1, 1)$$

- Special solutions.

$$\Big(R(t), F(t)\Big) = (ce^{t}, 0)$$

$$\Big(R(t), F(t)\Big) = (0, \bar{c}e^{-t})$$

Predator-Prey Systems

- Qualitative methods:
- phase plane (F vs R)

  at (R, F) plot $V(R,F) = \left(\frac{dR}{dt}, \frac{dF}{dt}\right)$
- get **slope field**
- normalize and get **direction field**
  - solutions follow the vector V.
- Numerical methods: use a method such as Euler's to approximate solution given an initial condition

Phase Plane

- Equilibrium Points.

$$\Big(R(t), F(t)\Big) = (0,0)$$

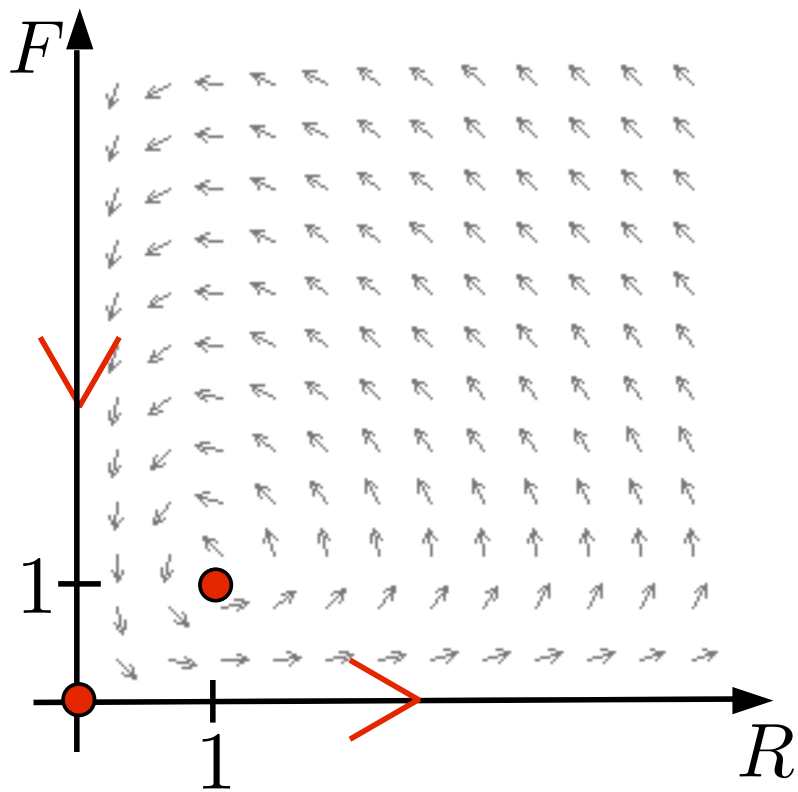
$$\Big(R(t), F(t)\Big) = (1,1)$$

- Special solutions.

$$\Big(R(t), F(t)\Big) = (ce^{t}, 0)$$

$$\Big(R(t), F(t)\Big) = (0, \bar{c}e^{-t})$$

- Direction field



General n-dim Systems

$$\begin{cases} \dfrac{dy_1}{dt} = F_1(y_1, y_2, ..., y_n, t) \\ \dfrac{dy_2}{dt} = F_2(y_1, y_2, ..., y_n, t) \\ \vdots \\ \dfrac{dy_n}{dt} = F_n(y_1, y_2, ..., y_n, t) \end{cases}$$

- Want to find the functions

$$y_1(t), y_2(t), ..., y_n(t)$$

2-dim Systems

$$\begin{cases} \dfrac{dy_1}{dt} = F_1(y_1, y_2, t) \\ \dfrac{dy_2}{dt} = F_2(y_1, y_2, t) \end{cases}$$

- Example

$$\begin{cases} \dfrac{dy_1}{dt} = y_1 + y_2 + t^2 \\ \dfrac{dy_2}{dt} = y_1 \cdot y_2 + sin(t) \end{cases}$$

Non-linear
Non-autonomous

2-dim Systems

- Example

$$\begin{cases} \dfrac{dy_1}{dt} = y_1 + y_2 \\ \dfrac{dy_2}{dt} = y_1 \cdot y_2 \end{cases}$$

Non-linear
Autonomous

- Example

$$\begin{cases} \dfrac{dy_1}{dt} = y_1 + y_2 \\ \dfrac{dy_2}{dt} = 2y_1 + y_2 \end{cases}$$

Linear
Autonomous

2-dim Systems

- Example

$$\begin{cases} \dfrac{dx}{dt} = 3x + 2y \\ \dfrac{dy}{dt} = 4x - (\sin t) \cdot y \end{cases}$$

Linear
Non-autonomous

- General form of linear ODE (2-dim)

$$\begin{cases} \dfrac{dx}{dt} = a_1(t) \cdot x + a_2(t) \cdot y + b_1(t) \\ \dfrac{dy}{dt} = a_3(t) \cdot x + a_4(t) \cdot y + b_2(t) \end{cases}$$

Vector Notation

$$\begin{cases} \dfrac{dy_1}{dt} = F_1(y_1, y_2, ..., y_n, t) \\ \quad\vdots \qquad\qquad\qquad \vdots \\ \dfrac{dy_n}{dt} = F_n(y_1, y_2, ..., y_n, t) \end{cases}$$

- Abbreviate

$$\frac{d\vec{\mathbf{Y}}}{dt} = \vec{\mathbf{Y}}' = F(\vec{\mathbf{Y}}, t) \qquad \vec{\mathbf{Y}} = \begin{pmatrix} y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{pmatrix}$$

Solutions

Solution of $\vec{\mathbf{Y}}' = F(\vec{\mathbf{Y}}, t)$

is a curve $\vec{\mathbf{Y}}(t) = \begin{pmatrix} y_1(t) \\ \cdot \\ \cdot \\ \cdot \\ y_n(t) \end{pmatrix}$

that "works"

- Equilibrium solution

$\vec{\mathbf{Y}}(t) =$ constant vector

$F(\vec{\mathbf{Y}}(t), t) = 0$ for all times t

Autonomous 2D system

$$\frac{d\vec{\mathbf{Y}}}{dt} = F(\vec{\mathbf{Y}}) \qquad \vec{\mathbf{Y}} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{aligned} x' &= f(x, y) \\ y' &= g(x, y) \end{aligned}$$

- Example $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix}$

$$\vec{\mathbf{Y}}' = \begin{pmatrix} y \\ -x \end{pmatrix} \text{ where } \vec{\mathbf{Y}}(t) = \begin{pmatrix} x \\ y \end{pmatrix}$$

- <u>Example</u>

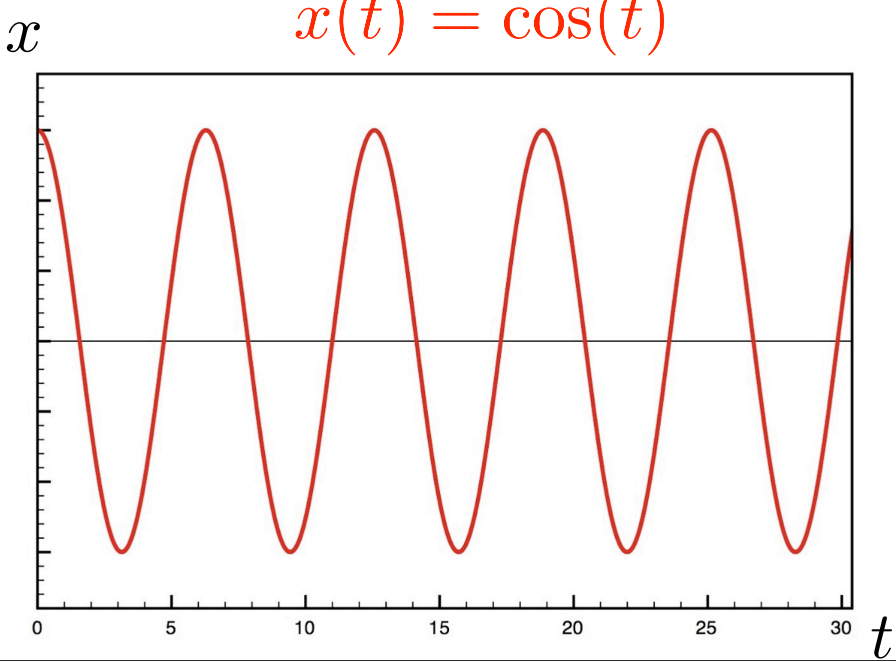
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix}$$

- <u>Try:</u> $x(t) = \cos(t)$

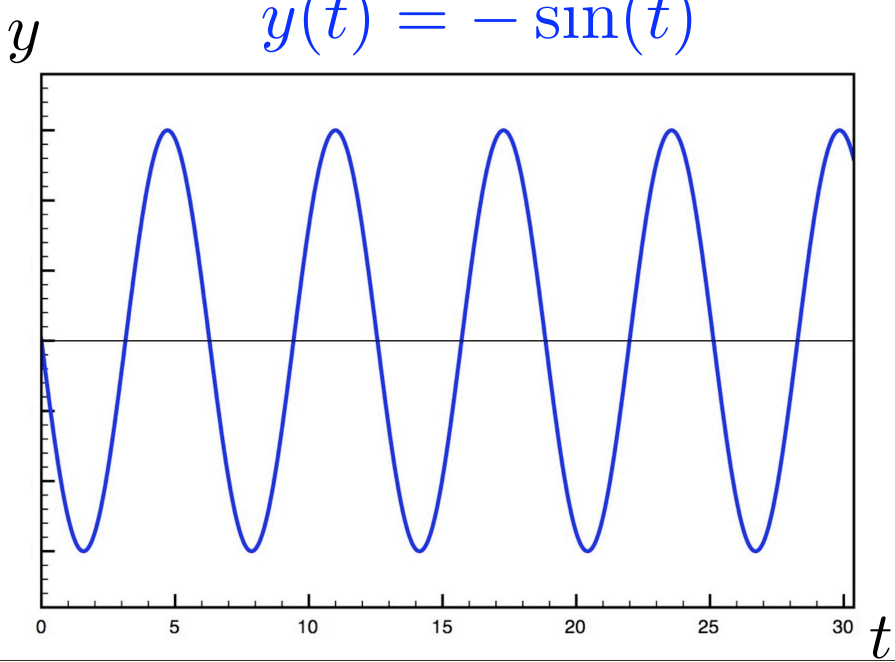
$$y(t) = -\sin(t)$$

$$\vec{\mathbf{Y}}(t) = \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix}$$

$x(t) = \cos(t)$

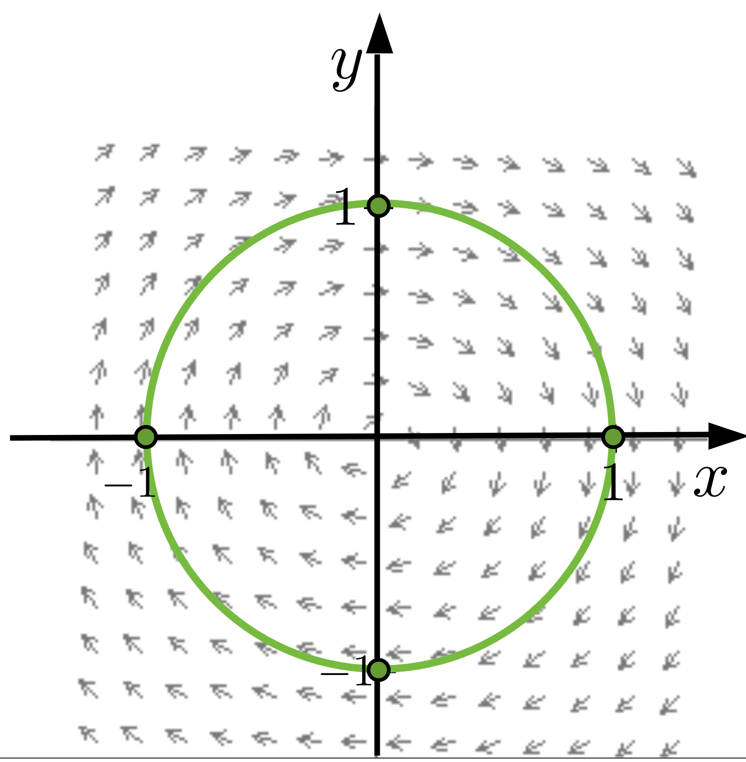


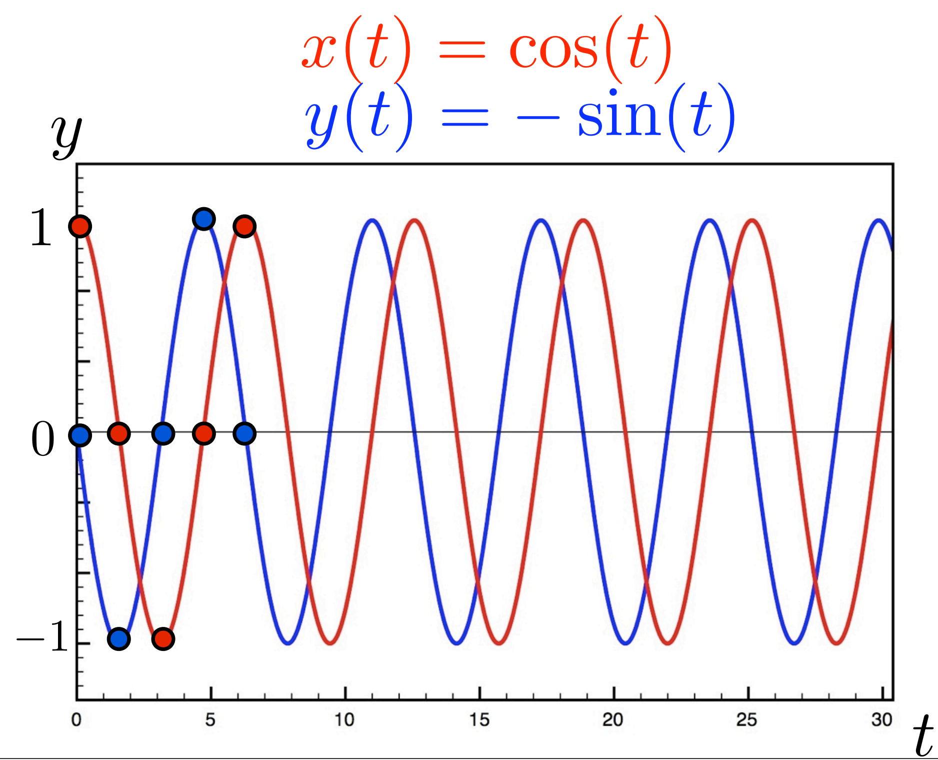
$y(t) = -\sin(t)$



- Example $$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix}$$

- Going from solutions graphs to phase plane





Decoupled Systems

- Completely decoupled

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -2x \\ -y \end{pmatrix}$$

- Partially decoupled

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3x - 4y \\ -2y \end{pmatrix}$$