Day 12: July 15th

- **Chapter: 3.2 Straight Line Solutions.**
  - Homework:
    - Page 271 #1-7 odd, 11, 13, 21, 23.

- **Chapter: 3.3 Phase Plane for Real E.V.**
  - Homework:
    - Page 287 #1-11 odd, 15, 19.

- Your New Best Friend: Linear 2-D Systems

$$\vec{\mathbf{Y}}' = \mathbf{A} \cdot \vec{\mathbf{Y}} \text{ where } \vec{\mathbf{Y}} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{cases} \dfrac{dx}{dt} = ax + by \\ \dfrac{dy}{dt} = cx + dy \end{cases}$$

- Linear 2-D Systems
  - Equilibrium points

$$\begin{cases} \dfrac{dx}{dt} = ax + by = 0 \quad (1) \\ \\ \dfrac{dy}{dt} = cx + dy = 0 \quad (2) \end{cases}$$

  - one solution: $x = y = 0$ others?

$$(1) \quad ax + by = 0 \quad \Rightarrow \quad x = -\frac{b}{a}y \quad (a \neq 0)$$

$$(2)\, c\Big(-\frac{b}{a}y\Big) + dy = 0 \;\Rightarrow\; \frac{ad - bc}{a} = 0$$

- <u>Linear 2-D Systems</u>

$$\frac{ad - bc}{a} \cdot y = 0 \Rightarrow \frac{ad - bc}{a} = 0 \text{ or } y = 0$$

$$ad - bc = 0 \qquad x = -\frac{b}{a} y = 0$$

- <u>Define</u>: Determinant

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

- If $\det \mathbf{A} \neq 0 \Rightarrow (x, y) = (0, 0)$ only eq point.

- Linear 2-D Systems
- If $\det \mathbf{A} = 0 \Rightarrow$ any y works

$$x = -\frac{b}{a}y \text{ straight line solution}$$

get a line of equilibrium points

- Example 1: $$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$
- Example 2: $$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Summary

$$\vec{\mathbf{Y}}' = \mathbf{A} \cdot \vec{\mathbf{Y}} \text{ where } \vec{\mathbf{Y}} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\det \mathbf{A} = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

- If $\det \mathbf{A} \neq 0 \Rightarrow (x, y) = (0, 0)$ only eq point.
- If $\det \mathbf{A} = 0$ straight line of equilibrium points.

- Strategy for solving 2-D systems

$$\vec{\mathbf{Y}}' = \mathbf{A} \cdot \vec{\mathbf{Y}} \text{ where } \vec{\mathbf{Y}} = \begin{pmatrix} x \\ y \end{pmatrix}$$

- find two sol $\vec{\mathbf{Y}}_1(t)$ and $\vec{\mathbf{Y}}_2(t)$ with

$$\vec{\mathbf{Y}}_1(0) \text{ and } \vec{\mathbf{Y}}_2(0)$$

linearly independent.

- then

$$k_1\vec{\mathbf{Y}}_1(t) + k_2\vec{\mathbf{Y}}_2(t)$$

is the general solution

- Why is this a general solution
  - Linear Independence:

if $\vec{\mathbf{V}}_1$ and $\vec{\mathbf{V}}_2$ are linearly independent

we can always find

k_1 and k_2

such that

$$k_1\vec{\mathbf{V}}_1 + k_2\vec{\mathbf{V}}_2 = \text{any vector}$$

- How do we find the two solutions
  - Linear Algebra

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

  - Characteristic Equation

$$\det \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} = 0$$

- Characteristic Equation:
  - second order equation in λ

$$\lambda^2 - \mathrm{tr}(\mathbf{A})\lambda + \det(\mathbf{A}) = 0$$

$$\det \mathbf{A} = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

$$\mathrm{tr}(\mathbf{A}) = a + d$$

- Eigenvalues: Roots of characteristic equation
- Example: $\begin{pmatrix} 2 & 1 \\ 0 & -2 \end{pmatrix}$

- Eigenvectors
    - vectors that satisfy

$$\mathbf{A} \cdot \vec{\mathbf{v}} = \lambda \vec{\mathbf{v}}$$

- How do we find eigenvectors

- Example: $\begin{pmatrix} 2 & 1 \\ 0 & -2 \end{pmatrix}$

Solutions

- if we have two distinct real eigenvalues λ_1, λ_2

$\vec{\mathbf{Y}}_1(t) = e^{\lambda_1 t} \cdot \vec{\mathbf{v}}_1$ is a solution

and

$\vec{\mathbf{Y}}_2(t) = e^{\lambda_2 t} \cdot \vec{\mathbf{v}}_2$ is a solution

$k_1\vec{\mathbf{Y}}_1(t) = k_1 e^{\lambda_1 t} \cdot \vec{\mathbf{v}}_1$ is also a solution

$k_2\vec{\mathbf{Y}}_2(t) = k_2 e^{\lambda_2 t} \cdot \vec{\mathbf{v}}_2$ as well

General Solution

- If $\vec{\mathbf{Y}}_1(0)$ and $\vec{\mathbf{Y}}_2(0)$ are linearly independent
- General Solution

$$\vec{\mathbf{Y}}(t) = k_1 \vec{\mathbf{Y}}_1(t) + k_2 \vec{\mathbf{Y}}_2(t)$$

$$= k_1 e^{\lambda_1 t} \cdot \vec{\mathbf{v}}_1 + k_2 e^{\lambda_2 t} \cdot \vec{\mathbf{v}}_2$$