# Day 13: July 19th

- **Chapter: 3.3 Phase Plane for Real E.V.**

- **Chapter: 3.4 Complex Eigenvalues (E.V.)**
- Homework:
  - Page 304 #1-15 odd, 23, 25.

# Review

$$\vec{\mathbf{Y}}' = \mathbf{A} \cdot \vec{\mathbf{Y}} \text{ where } \vec{\mathbf{Y}} = \begin{pmatrix} x \\ y \end{pmatrix}$$

- Equilibrium points
  - If $\det \mathbf{A} \neq 0 \Rightarrow (x, y) = (0, 0)$ only eq point.
  - If $\det \mathbf{A} = 0$ straight line of equilibrium points.
- Characteristic Equation

$$\det \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} = 0$$

- Eigenvalues roots of characteristic equation

# Review

- Eigenvectors

$$\boxed{\mathbf{A} \cdot \vec{\mathbf{v}} = \lambda \vec{\mathbf{v}}}$$

- Finding eigenvectors
  - Non-zero solutions of

$$(\mathbf{A} - \lambda \mathbf{I}) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

  - always get a redundant system of equations

# General Solution

- if we have two distinct real eigenvalues: $\lambda_1, \lambda_2$
  and can find two distinct eigenvectors: $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2$

$$\left.\begin{aligned}\vec{\mathbf{Y}}_1(t) &= e^{\lambda_1 t} \cdot \vec{\mathbf{v}}_1 \\ \vec{\mathbf{Y}}_2(t) &= e^{\lambda_2 t} \cdot \vec{\mathbf{v}}_2\end{aligned}\right\}$$ two straight line solutions

- If $\vec{\mathbf{Y}}_1(0)$ and $\vec{\mathbf{Y}}_2(0)$ are linearly independent

$$\boxed{\begin{aligned}\vec{\mathbf{Y}}(t) &= k_1 \vec{\mathbf{Y}}_1(t) + k_2 \vec{\mathbf{Y}}_2(t) \\ &= k_1 e^{\lambda_1 t} \cdot \vec{\mathbf{v}}_1 + k_2 e^{\lambda_2 t} \cdot \vec{\mathbf{v}}_2\end{aligned}}$$

# Last Lecture

$$\vec{\mathbf{Y}}' = \begin{pmatrix} 2 & 1 \\ 0 & -2 \end{pmatrix} \vec{\mathbf{Y}}$$

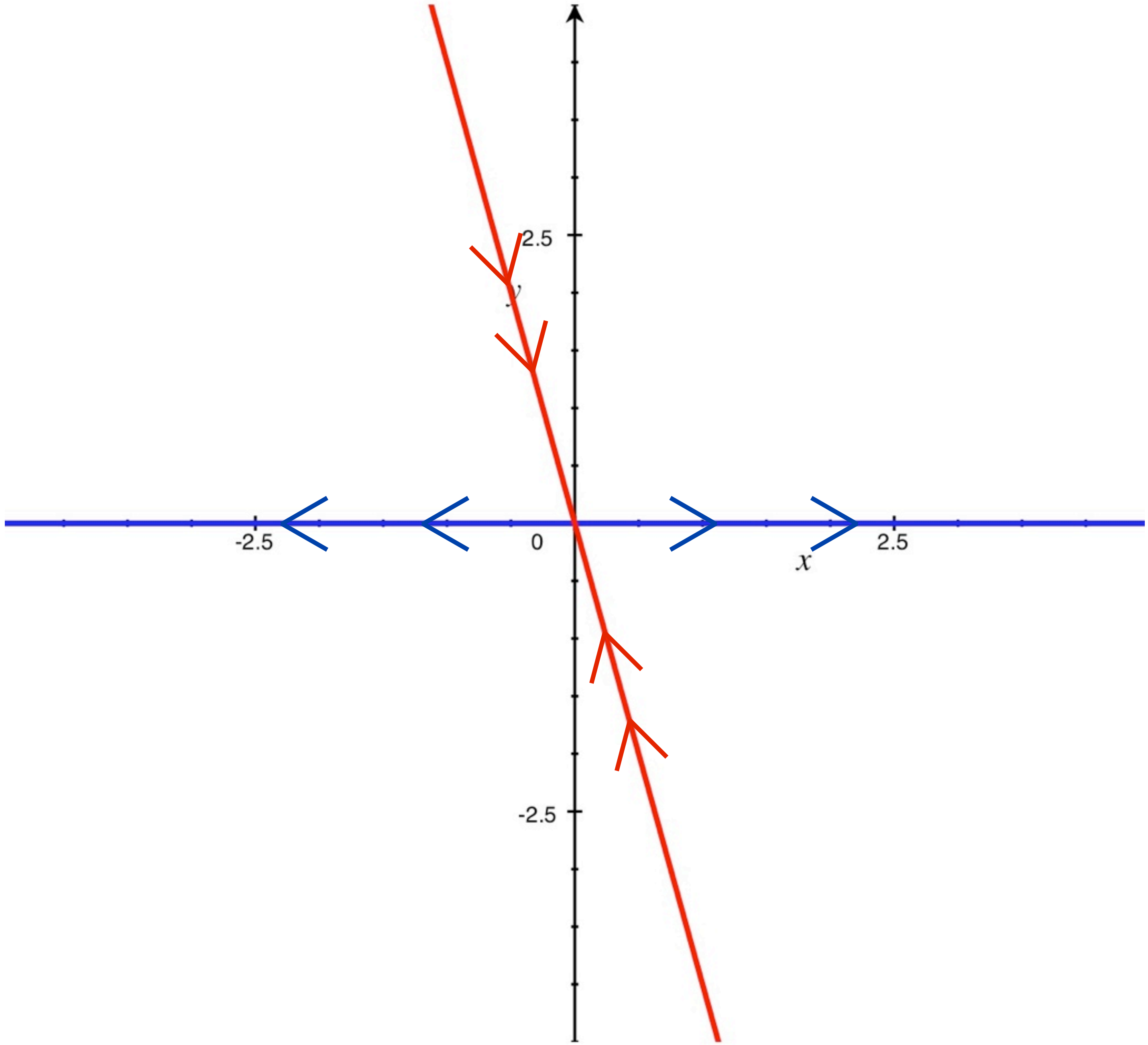
- Eigenvalues: $\lambda_1 = 2, \lambda_2 = -2$
- Eigenvectors:

$$\vec{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \vec{\mathbf{v}}_2 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

- General Solution:

$$\vec{\mathbf{Y}}(t) = k_1 e^{\lambda_1 t} \cdot \vec{\mathbf{v}}_1 + k_2 e^{\lambda_2 t} \cdot \vec{\mathbf{v}}_2$$

$$\vec{\mathbf{Y}}(t) = k_1 e^{2t} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 e^{-2t} \cdot \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$


2.5
y
-2.5
0
2.5
x
-2.5

# Last Lecture

$$\vec{\mathbf{Y}}(t) = k_1 e^{2t} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 e^{-2t} \cdot \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

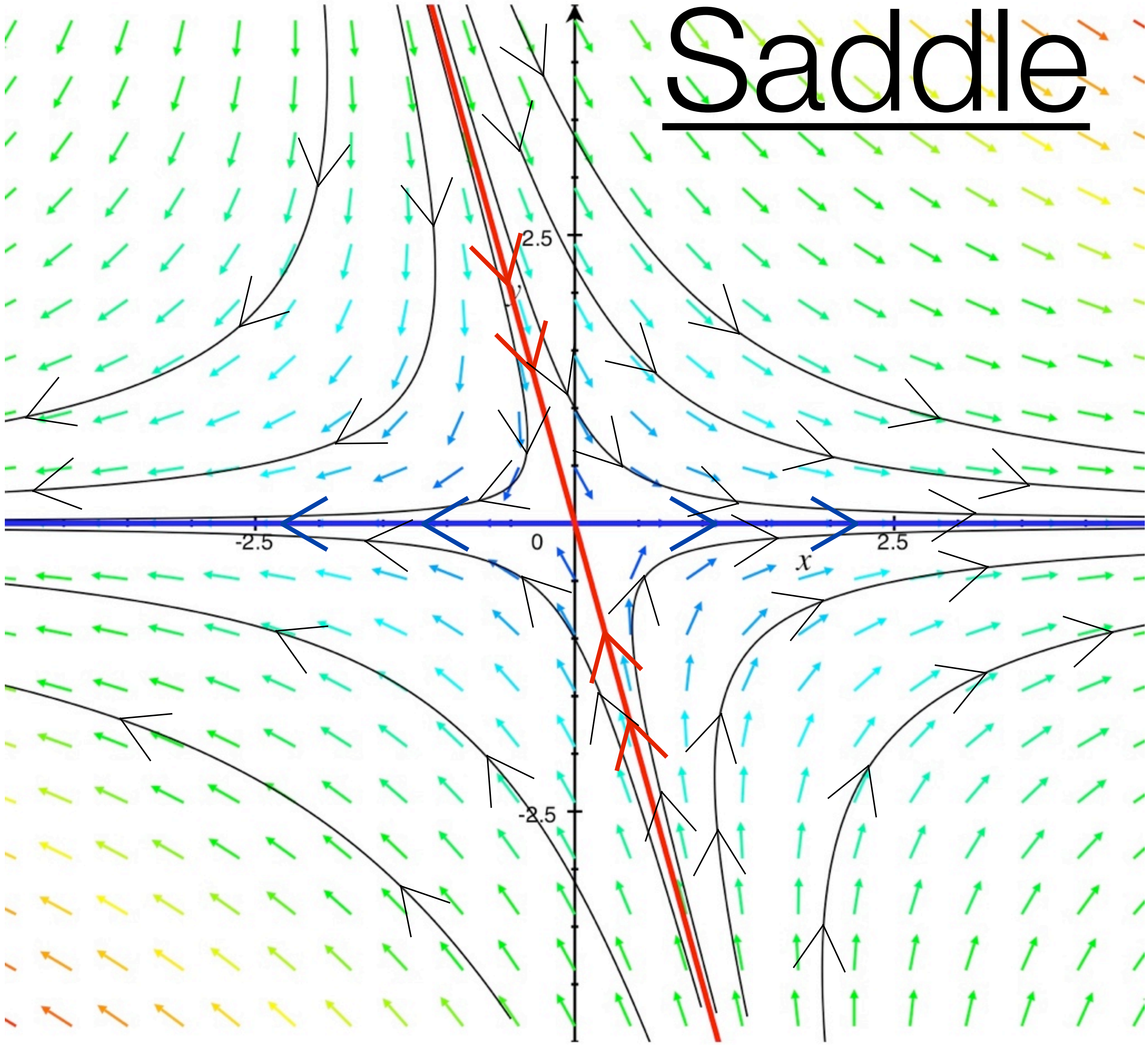
$$\vec{\mathbf{Y}}(t) = \begin{pmatrix} \cancel{k_1} e^{2t} + \cancel{k_2} e^{-2t} \\ -2\cancel{k_2} e^{-2t} \end{pmatrix}$$

- Long term behavior: $t \to \infty$ and $t \to -\infty$

$$\vec{\mathbf{Y}}(t) \sim \begin{pmatrix} k_1 e^{2t} \\ 0 \end{pmatrix} = k_1 e^{2t} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ as } t \to \infty$$

$$\vec{\mathbf{Y}}(t) \sim \begin{pmatrix} k_2 e^{-2t} \\ -2k_2 e^{-2t} \end{pmatrix} = k_2 e^{-2t} \cdot \begin{pmatrix} 1 \\ -2 \end{pmatrix} \text{ as } t \to -\infty$$

# Saddle



# Nifty Trick

If $\mathbf{A}$ triangular matrix

$$\mathbf{A} = \begin{pmatrix} a & \# \\ 0 & b \end{pmatrix}$$

or

$$\mathbf{A} = \begin{pmatrix} a & 0 \\ \# & b \end{pmatrix}$$

eigenvalues are: $a, b$

# Example

$$\vec{\mathbf{Y}}' = \begin{pmatrix} -1 & -1 \\ 0 & -3 \end{pmatrix} \vec{\mathbf{Y}}$$

- Eigenvalues:

$$\lambda_1 = -3, \lambda_2 = -1$$

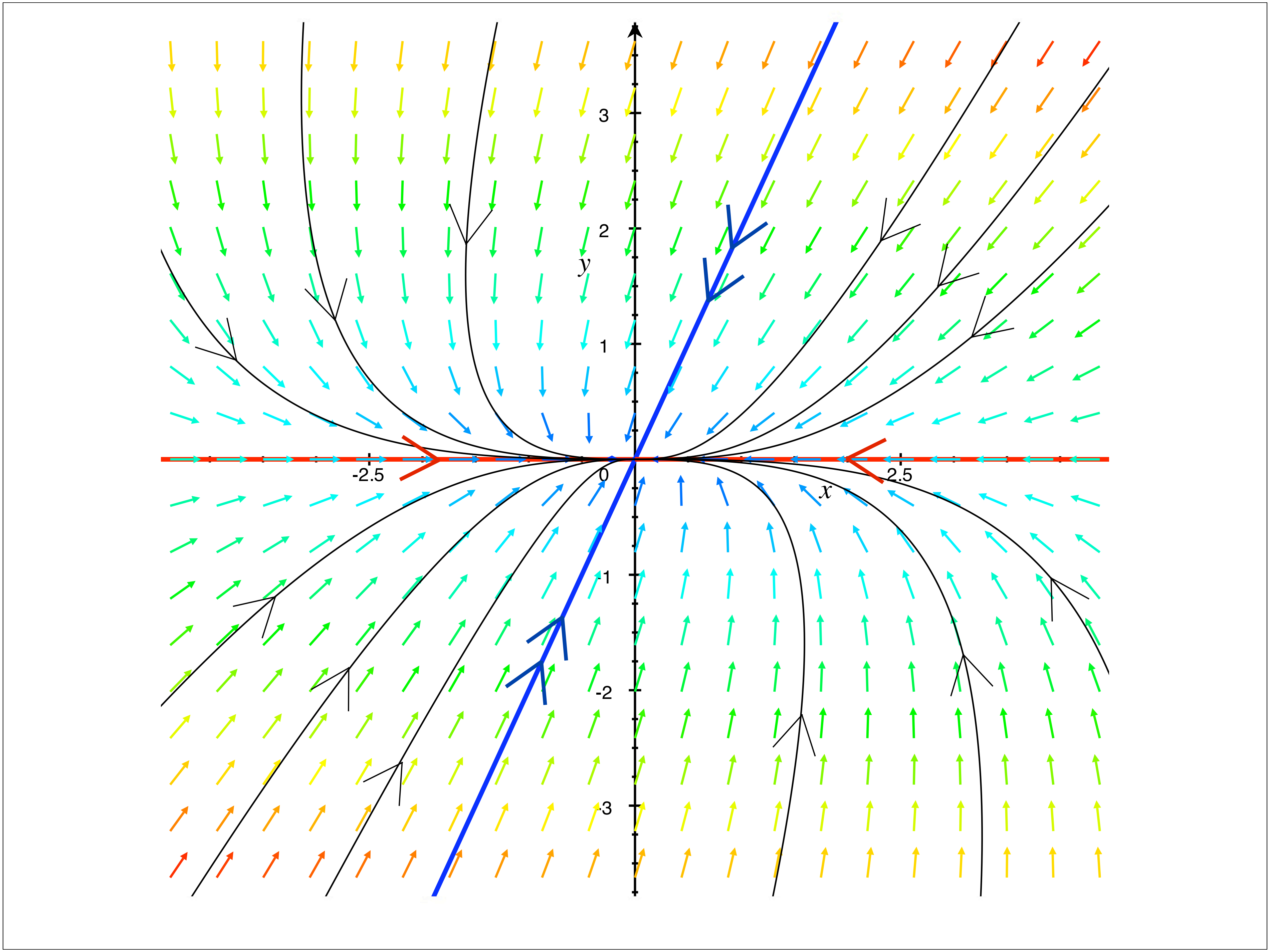
- Eigenvectors:

$$\vec{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \vec{\mathbf{v}}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- General Solution:

$$\vec{\mathbf{Y}}(t) = k_1 e^{\lambda_1 t} \cdot \vec{\mathbf{v}}_1 + k_2 e^{\lambda_2 t} \cdot \vec{\mathbf{v}}_2$$

$$\vec{\mathbf{Y}}(t) = k_1 e^{-3t} \cdot \begin{pmatrix} 1 \\ -2 \end{pmatrix} + k_2 e^{-t} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



3
2
1
0
-1
-2
-3
-2.5
2.5
y
x

- Two distinct negative eigenvalues: $\lambda_1, \lambda_2$

faster-stronger

$$\lambda_2 < \lambda_1 < 0$$

weaker-slower

- General solution:

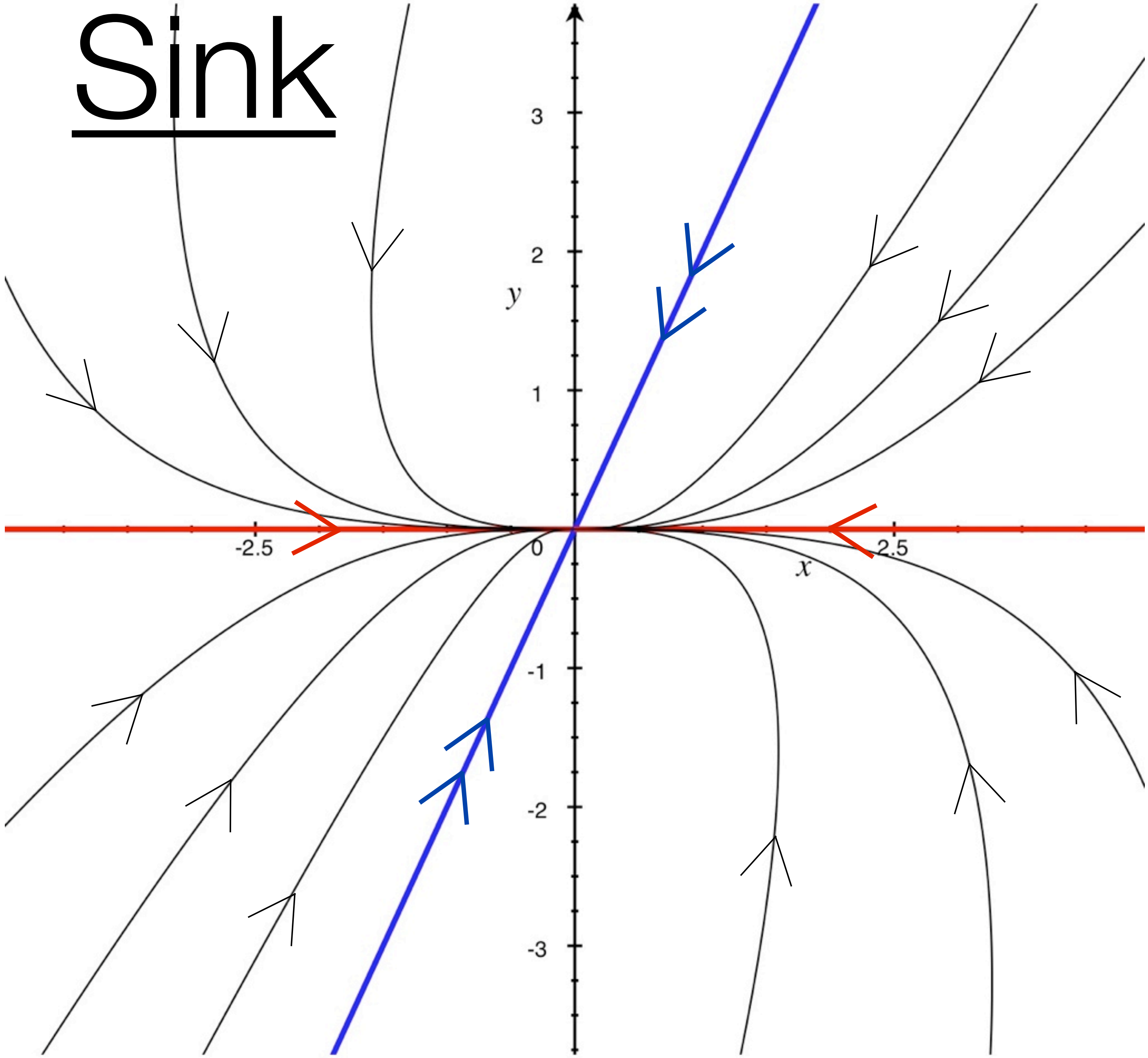
$$\vec{\mathbf{Y}}(t) = k_1 e^{\lambda_1 t} \cdot \vec{\mathbf{v}}_1 + k_2 e^{\lambda_2 t} \cdot \vec{\mathbf{v}}_2$$

- Long term behavior: $t \to \infty$ and $t \to -\infty$

| |
|---|
| $t \longrightarrow \infty$: approaches origin tangent to the weaker eigenvalue straight line solution. |

| |
|---|
| $t \longrightarrow -\infty$: moves away from the origin parallel to the stronger straight line solution. |

# Sink



# Example

$$\vec{\mathbf{Y}}' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \vec{\mathbf{Y}}$$

- Eigenvalues:

$$\lambda_1 = 3, \lambda_2 = 1$$

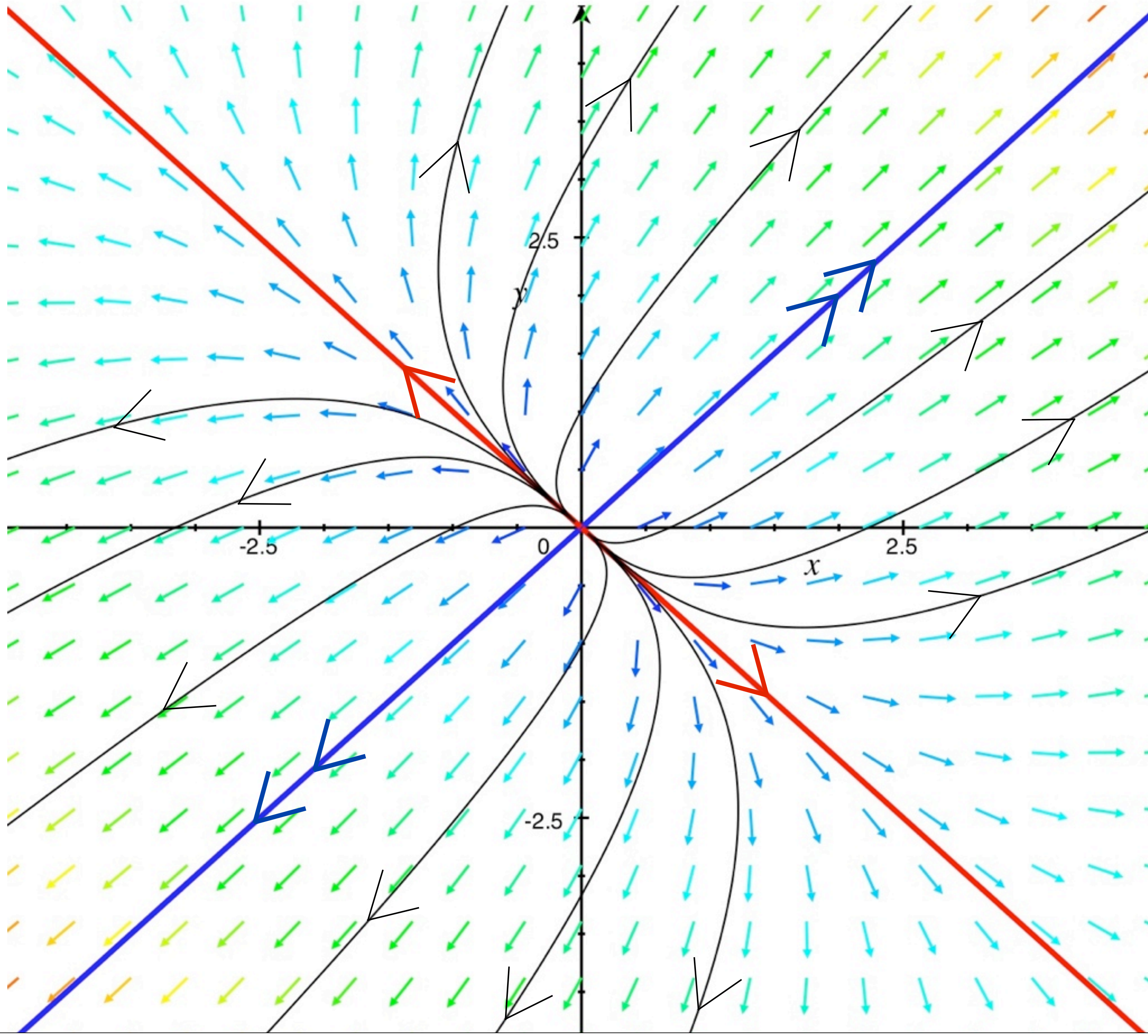
- Eigenvectors:

$$\vec{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \vec{\mathbf{v}}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

- General Solution:

$$\vec{\mathbf{Y}}(t) = k_1 e^{\lambda_1 t} \cdot \vec{\mathbf{v}}_1 + k_2 e^{\lambda_2 t} \cdot \vec{\mathbf{v}}_2$$

$$\vec{\mathbf{Y}}(t) = k_1 e^{3t} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + k_2 e^{t} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$



y
x
2.5
-2.5
0
2.5
-2.5

- Two distinct positive eigenvalues: $\lambda_1, \lambda_2$

slower-weaker

$$0 < \lambda_1 < \lambda_2$$

stronger-faster

- General solution

$$\vec{\mathbf{Y}}(t) = k_1 e^{\lambda_1 t} \cdot \vec{\mathbf{v}}_1 + k_2 e^{\lambda_2 t} \cdot \vec{\mathbf{v}}_2$$

- Long term behavior: $t \to \infty$ and $t \to -\infty$

$t \to \infty$: moves away from the origin parallel to the stronger straight line solution.

$t \to -\infty$: approaches origin tangent to the weaker eigenvalue straight line solution.

# Source