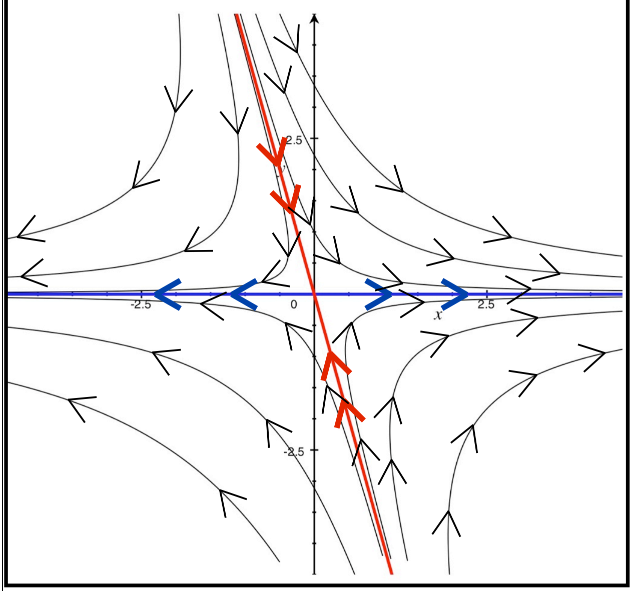
Day 14: July 20th

- **Chapter: 3.4 Complex Eigenvalues (E.V.)**
- Homework:
  - Page 304 #1-15 odd, 23, 25.
- **Chapter: 3.5 Repeated Eigenvalues (E.V.)**

Review

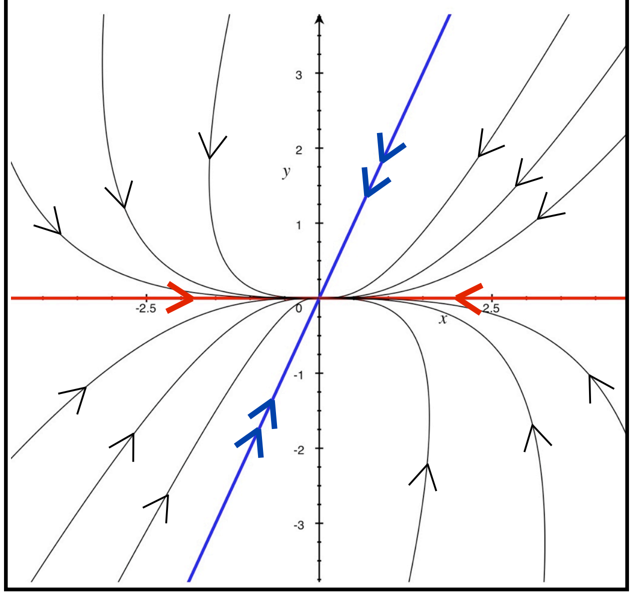
Saddle

$$\lambda_1 < 0 < \lambda_2$$



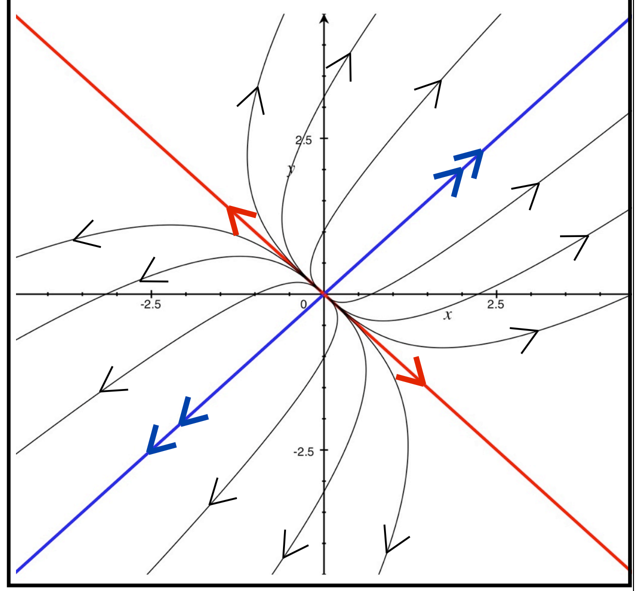
Sink

$$\lambda_2 < \lambda_1 < 0$$



Source

$$0 < \lambda_1 < \lambda_2$$



$$\lambda_1, \lambda_2 \in \mathbb{R}$$

Complex Eigenvalues

- Roots of characteristic equation, λ_1, λ_2 can be complex:
- Example:

$$\vec{\mathbf{Y}}' = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \vec{\mathbf{Y}}$$

- Trick: Characteristic Equation

$$\lambda^2 - \mathrm{tr}(\mathbf{A})\lambda + \det(\mathbf{A}) = 0$$

Where: $\det(\mathbf{A}) = ad - bc$

$$\mathrm{tr}(\mathbf{A}) = a + d$$

Complex Eigenvalues

- Eigenvalue λ_1 and eigenvector $\vec{\mathbf{v}}_1$
- One solution:

$$\vec{\mathbf{Y}}_1(t) = k_1 \underbrace{e^{\lambda_1 t}}_{\downarrow} \cdot \vec{\mathbf{v}}_1$$

$e^{(1+2\mathrm{i})t}$ - what does this mean?

- Euler's Formula: $\boxed{e^{\mathrm{i}t} = \cos(t) + \mathrm{i}\sin(t)}$

$t = \pi$: $$e^{\mathrm{i}\pi} = \cos(\pi) + \mathrm{i}\sin(\pi) = -1$$

$$e^{\mathrm{i}\pi} + 1 = 0$$ GOD equation

- <u>Euler's Formula</u>: $\boxed{e^{\mathrm{i}t} = \cos(t) + \mathrm{i}\sin(t)}$

$$e^{(1+2\mathrm{i})t} = e^t \cdot e^{2\mathrm{i}t} = e^t \cdot e^{(2t)\mathrm{i}}$$

$$= e^t \cdot \big(\cos(2t) + \mathrm{i}\sin(2t)\big)$$

- <u>Example</u>:

$$e^{-\mathrm{i}t} = e^{\mathrm{i}(-t)} = \cos(-t) + \mathrm{i}\sin(-t)$$

$$= \cos(t) - \mathrm{i}\sin(t)$$

General Solution

- One solution:

$$\vec{\mathbf{Y}}_1(t) = e^{\lambda_1 t} \cdot \vec{\mathbf{v}}_1$$

break up solution

$\vec{\mathbf{Y}}_{\mathrm{Re}}(t)$ — Real part of solution

$\vec{\mathbf{Y}}_{\mathrm{Im}}(t)$ — Imaginary part of solution

- Fact: Both $\vec{\mathbf{Y}}_{\mathrm{Re}}(t)$ and $\vec{\mathbf{Y}}_{\mathrm{Im}}(t)$ are solutions.
- If $\vec{\mathbf{Y}}_{\mathrm{Re}}(t)$ and $\vec{\mathbf{Y}}_{\mathrm{Im}}(t)$ are linearly independent

Gen sol: $\vec{\mathbf{Y}}(t) = k_1 \vec{\mathbf{Y}}_{\mathrm{Re}}(t) + k_2 \vec{\mathbf{Y}}_{\mathrm{Im}}(t)$

Back to example

$$\vec{\mathbf{Y}}' = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \vec{\mathbf{Y}}$$

- Eigenvalues: $1 \pm 2\mathrm{i}$
- Eigenvectors:

  pick either eigenvalue: $\lambda = 1 + 2\mathrm{i}$

$$\vec{\mathbf{v}} = \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix}$$

- One Solution: $\vec{\mathbf{Y}}(t) = e^{\lambda t}\vec{\mathbf{v}} = e^{(1+2\mathrm{i})t} \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix}$

- <u>Break up:</u> $$e^{(1+2\mathrm{i})t}\begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix}$$

$$e^{t}\cdot\begin{pmatrix} \cos(2t) + \mathrm{i}\sin(2t) \\ -\sin(2t) + \mathrm{i}\cos(2t) \end{pmatrix}$$

$$\vec{\mathbf{Y}}_{\mathrm{Re}}(t) = e^{t}\cdot\begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix}$$

$$\vec{\mathbf{Y}}_{\mathrm{Im}}(t) = e^{t}\cdot\begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix}$$

NOTE: There is no i in $\vec{\mathbf{Y}}_{\mathrm{Im}}(t)$

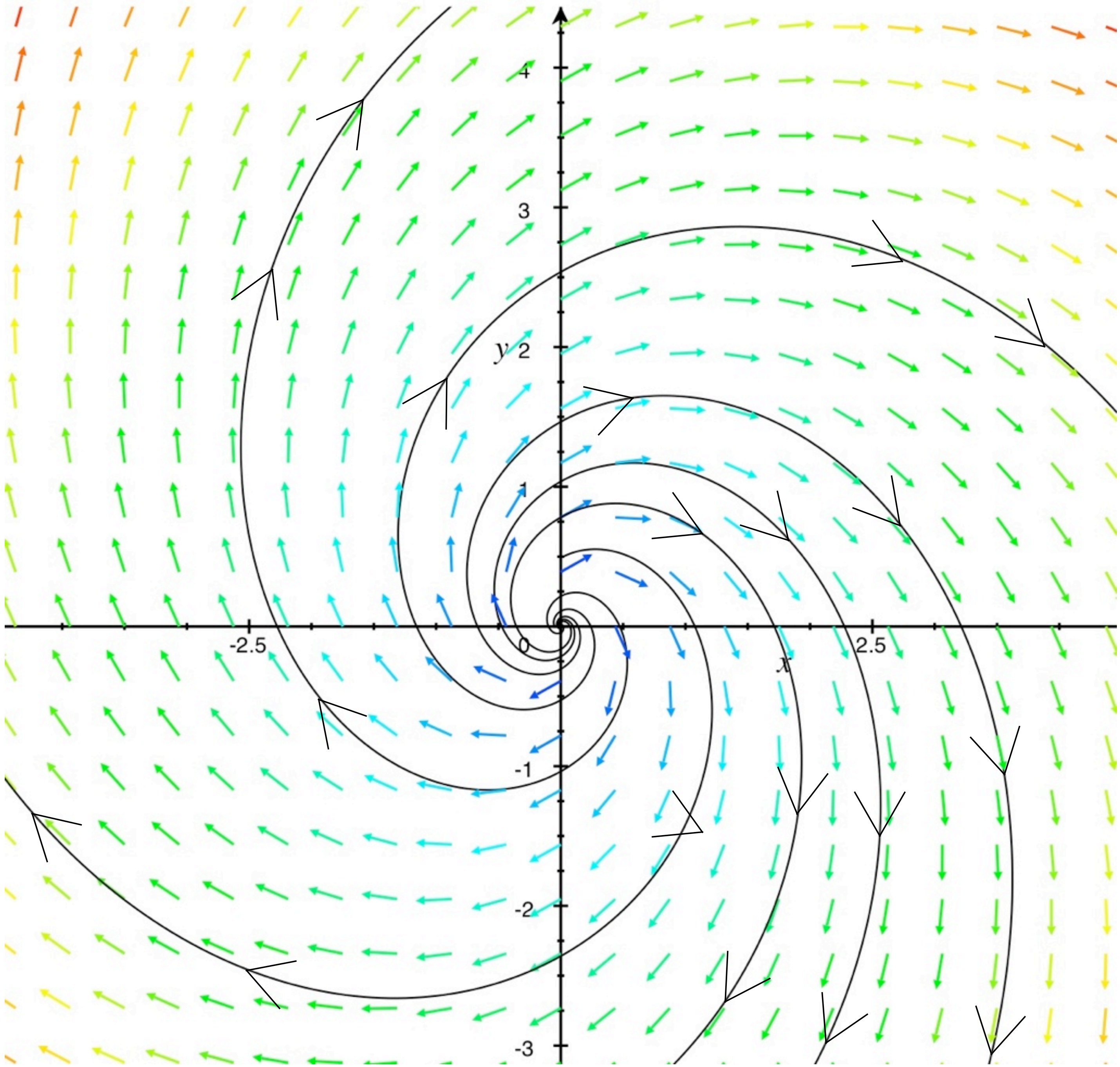
$\vec{\mathbf{Y}}_{\mathrm{Re}}(0)$ linearly independent $\vec{\mathbf{Y}}_{\mathrm{Im}}(0)$?

$$\vec{\mathbf{Y}}_{\mathrm{Re}}(0) = e^0 \cdot \begin{pmatrix} \cos(0) \\ -\sin(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\vec{\mathbf{Y}}_{\mathrm{Im}}(0) = e^0 \cdot \begin{pmatrix} \sin(0) \\ \cos(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Gen sol: $\vec{\mathbf{Y}}(t) = k_1 \vec{\mathbf{Y}}_{\mathrm{Re}}(t) + k_2 \vec{\mathbf{Y}}_{\mathrm{Im}}(t)$

$$= k_1 e^t \begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix} + k_2 e^t \begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix}$$



y
x
4
3
2
1
-1
-2
-3
-2.5
0
2.5

Example

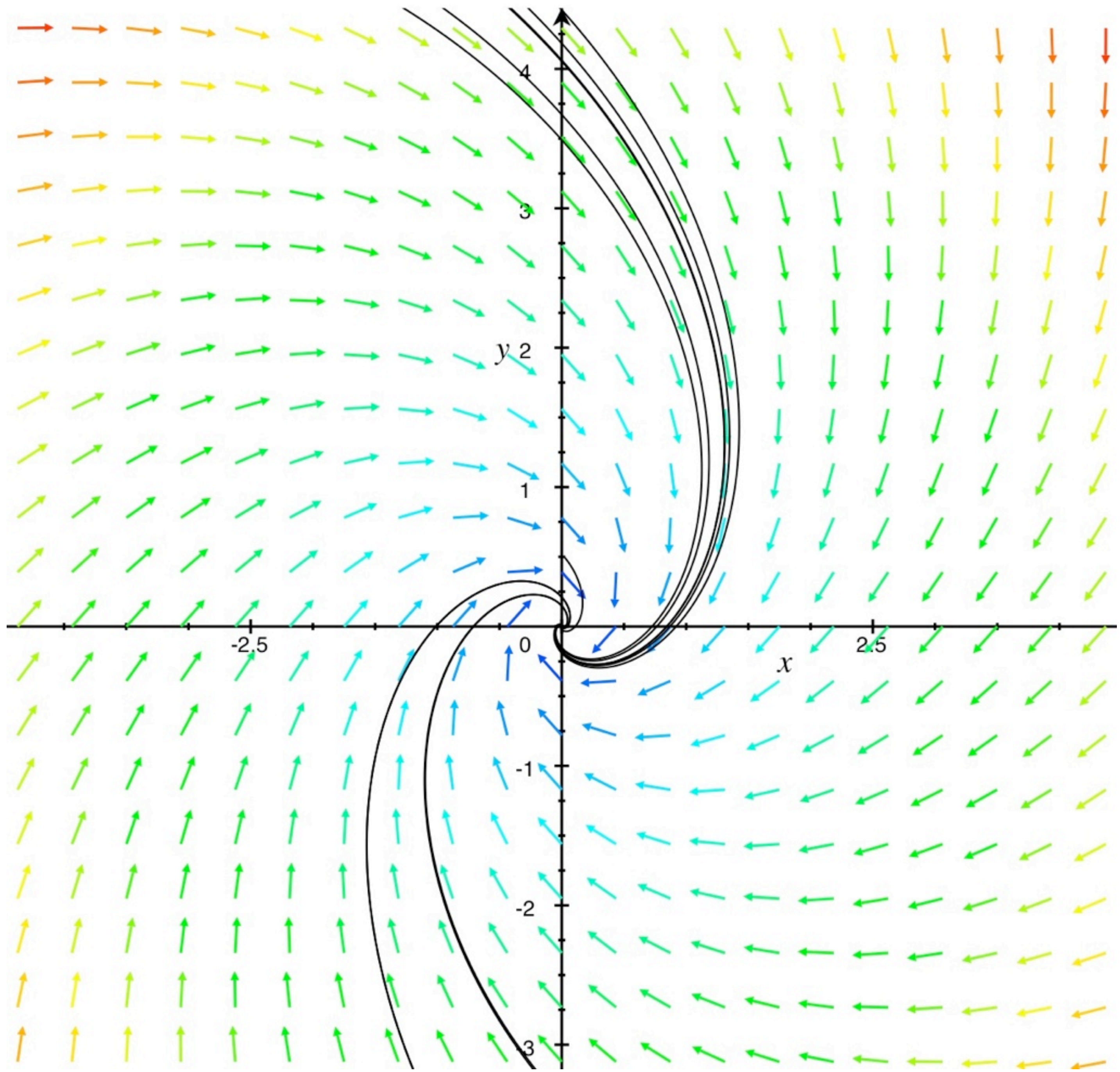
$$\vec{\mathbf{Y}}' = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \vec{\mathbf{Y}}$$

- Eigenvalues: $-1 \pm \mathrm{i}$
- Eigenvectors:

  pick either eigenvalue: $\lambda = -1 + \mathrm{i}$

$$\vec{\mathbf{v}} = \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix}$$

- One Solution: $\vec{\mathbf{Y}}(t) = e^{\lambda t}\vec{\mathbf{v}} = e^{(-1+\mathrm{i})t} \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix}$


4
3
y 2
1
-2.5
0
2.5
x
-1
-2
-3

Complex Eigenvalues

- Get a pair of complex eigenvalues: $a \pm \mathrm{i}b$
- $a > 0$ spiral source
- $a < 0$ spiral sink
- $a = 0$ (spiral) center