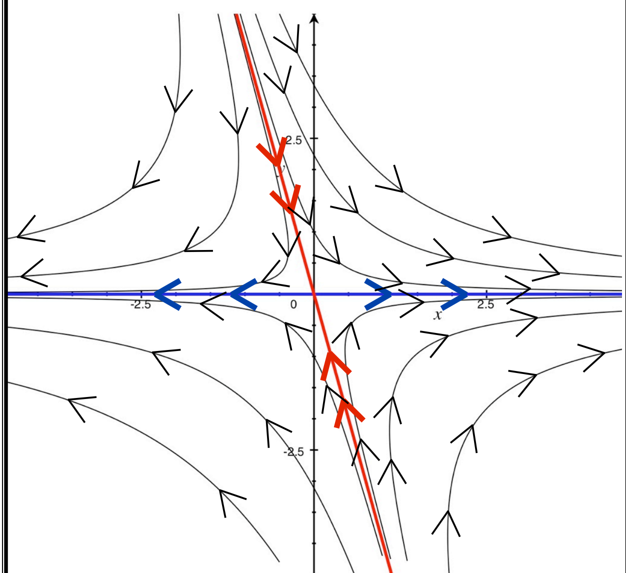
Day 15: July 21st

- **Chapter: 3.5 Repeated Eigenvalues (E.V.)**
  - Homework:
    - Page 321 #1-7 odd, 11, 21, 23.
- **Chapter: 3.6 Second Order Linear Equations**
- **Midterm 2 on Thursday, July 22nd:**
  Chapter 2 (2.1-2.3) and Chapter 3 (3.1-3.4)
- **LAB 2 posted due Wed. July 28th**

Review: Real Distinct Eigenvalues

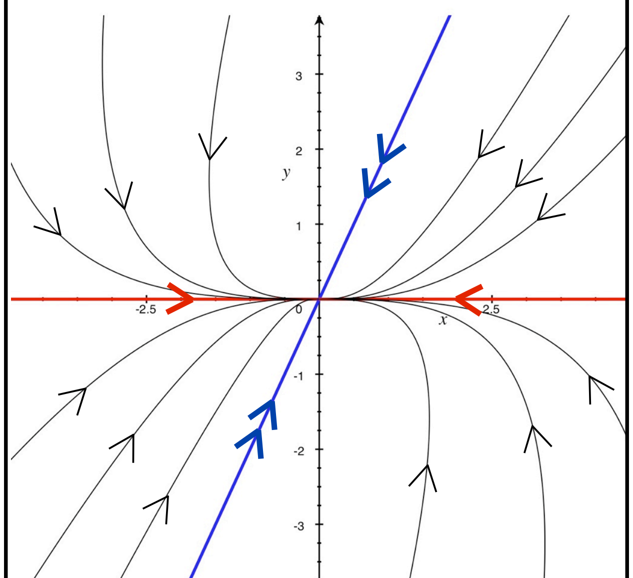
Saddle

$$\lambda_1 < 0 < \lambda_2$$



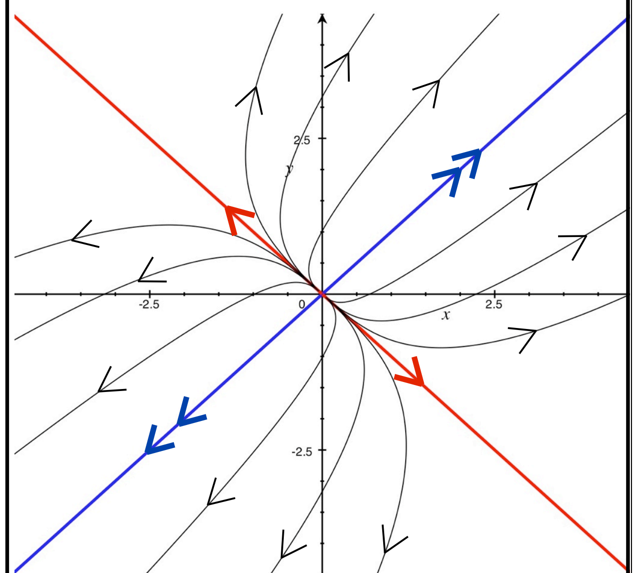
Sink

$$\lambda_2 < \lambda_1 < 0$$



Source

$$0 < \lambda_1 < \lambda_2$$



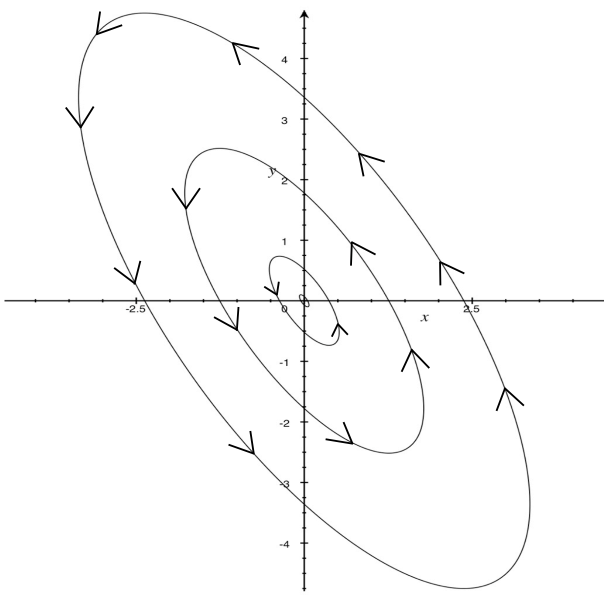
$$\lambda_1, \lambda_2 \in \mathbb{R}$$

Review: Complex Eigenvalues

$$\lambda = a \pm \mathrm{i}b$$

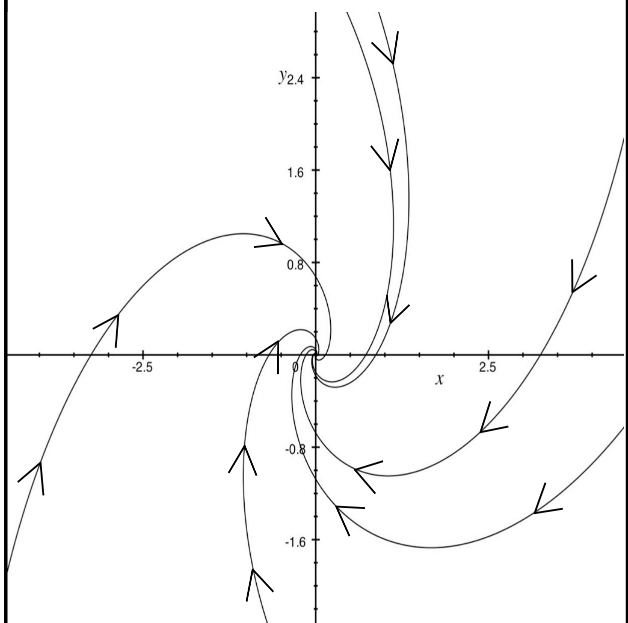
Center

$$a = 0$$



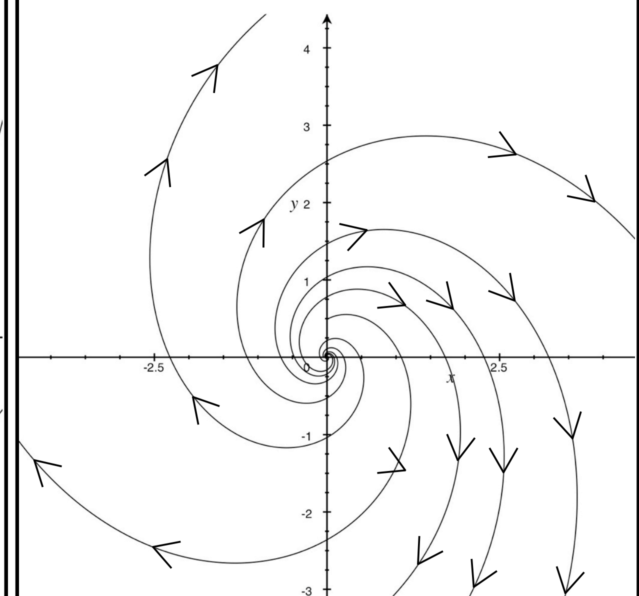
Spiral Sink

$$a < 0$$



Spiral Source

$$a > 0$$



In all cases rotation can be clockwise or counter clockwise.

Repeated Eigenvalues

- Example:

$$\vec{\mathbf{Y}}' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \vec{\mathbf{Y}}$$

- Eigenvalue: $\lambda = 1$ (repeated)

- Eigenvector:

$$\vec{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- One Solution: $\vec{\mathbf{Y}}_1(t) = k_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

How do we find another solution?

Repeated Eigenvalues

- Example:

$$\vec{\mathbf{Y}}' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \vec{\mathbf{Y}}$$

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$y' = y$$

$$y(t) = k_1 e^t$$

$$x' = x + y = x + k_1 e^t$$

- Example continued:

$$x' = x + k_1 e$$

Guess: $x = Ce^t$

Guess: $x = Cte^t$

- General solution: $x(t) = k_1 te^t + k_2 e^t$

- Altogether:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$x(t) = k_1 t e^t + k_2 e^t$$

$$y(t) = k_1 e^t$$

$$\vec{\mathbf{Y}}(t) = t e^t \begin{pmatrix} k_1 \\ 0 \end{pmatrix} + e^t \begin{pmatrix} k_2 \\ k_1 \end{pmatrix}$$

$$\mathbf{Y}(t) = k_1 t e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^t \begin{pmatrix} k_2 \\ k_1 \end{pmatrix}$$

eigenvector $\vec{\mathbf{v}}_1$ what is this?

Strategy for second solution

- Repeated eigenvalue:
- Find eigenvector:
- Solve $\boxed{(\mathbf{A} - \lambda \mathbf{I})\vec{\mathbf{Y}}_2 = \vec{\mathbf{Y}}_1 \text{ for } \vec{\mathbf{Y}}_2}$ $(\star)$
- Then if $\vec{\mathbf{Y}}_1$ and $\vec{\mathbf{Y}}_2$ are linearly independent
- <u>**General solution**</u>:

$$\vec{\mathbf{Y}}(t) = \boxed{k_1 e^{\lambda t} \cdot \vec{\mathbf{Y}}_1 + k_2 \left(t e^{\lambda t} \vec{\mathbf{Y}}_1 + e^{\lambda t} \vec{\mathbf{Y}}_2\right)}$$

str. lin. sol. λ str. lin. sol. λ solves $(\star)$

- Keep in mind that $\vec{\mathbf{Y}}_2$ is not an eigenvector.

Example

$$\vec{\mathbf{Y}}' = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix} \vec{\mathbf{Y}}$$

- Eigenvalue: $\lambda = 2$
- Eigenvector: $\vec{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$
- Straight line solution: $\vec{\mathbf{Y}}_1 = e^{2t} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$
- Second solution?

- Example continued:

$$(\mathbf{A}-\lambda\mathbf{I})\vec{\mathbf{Y}}_2=\vec{\mathbf{Y}}_1$$

- One solution:

$$\vec{\mathbf{Y}}_1=e^{2t}\cdot\begin{pmatrix}1\\-1\end{pmatrix}$$

$$\vec{\mathbf{Y}}_2=te^{2t}\cdot\begin{pmatrix}1\\-1\end{pmatrix}+e^{2t}\cdot\begin{pmatrix}1\\0\end{pmatrix}$$

- General solution: $\vec{\mathbf{Y}}(t)=k_1\vec{\mathbf{Y}}_1+k_2\vec{\mathbf{Y}}_2$

Phase Plane

Summary

- Real Distinct Eigenvalues
- Complex Eigenvalues
- Repeated Eigenvalues
- One straight line solution:

others

leave tangentially

or

come in tangentially

Second Order Linear Equations

- Easier way to solve second order ODEs
  - guess: e^{st}
- Example: $y^{''} + 3y^{'} + 2y = 0$
  - try: e^{st}
  - solutions: e^{-2t} and e^{-t}
  - so are: αe^{-2t} and βe^{-t}
  - also: $\alpha e^{-2t} + \beta e^{-t}$
- Homework: $y^{''} + 5y^{'} + 6y = 0$

- Easier way to solve second order ODEs
  - guess: e^{st}
- Example: $y^{''} + 2y^{'} + 1y = 0$
  - try: e^{st}
  - one solution: e^{-t}
  - where is the other?
  - guess: $t \cdot e^{-t}$ YES!
  - General Solution: $k_1 e^{-t} + k_2 \cdot t \cdot e^{-t}$
  - Homework: $y^{''} = 0$