Day 18: July 28th

- **Chapter: 4.2 Sinusoidal Forcing**
  - Homework:
    - Page 406 #1, 5, 9, 11, 13, 17.
- **Chapter: 4.3 Undamped Forcing & Resonance**
  - Homework:
    - Page 418 #1, 3, 9, 21.
    - Chapter 4: Review Problems: Page 443 #1, 2, 4, 11, 15, 17, 23, 25.
- **LAB 3 posted, due Aug. 4th**

- Strategy for solving: $y'' + by' + ky = 0$

  guess: $y = e^{st}$

  $$s^2 + bs + k = 0$$

  find roots: s_1, s_2

- Case 1

  $s_1, s_2 \in \mathbb{R} \quad s_1 \neq s_2$

  get solutions: $y_1(t) = e^{s_1 t} \quad y_2(t) = e^{s_2 t}$

  general solution: $y(t) = k_1 e^{s_1 t} + k_2 e^{s_2 t}$

- Case 2: complex roots: $s = \alpha \pm \mathrm{i}\beta$
  - Formula for general solution:

$$y(t) = e^{\alpha t}(k_1 \cos(\beta t) + k_2 \sin(\beta t))$$

- Case 3: repeated roots: s_1

  get solutions: $y_1(t) = e^{s_1 t}$ $y_2(t) = t \cdot e^{s_1 t}$

  general solution: $y(t) = k_1 e^{s_1 t} + k_2 t e^{s_1 t}$
  - Examples

$$y'' + 4y' + 4y = 0$$

Forced Harmonic Oscillator

$$y'' + by' + ky = F(t)$$ forcing term

often periodic

$$y'' + by' + ky = A \cdot \cos(\omega t)$$

second order NH equation

recall first order case

$$y' + ay = F(t) \qquad \text{(NH)}$$

$$y' + ay = 0 \qquad \text{(H)}$$

general solution for (H): $y(t) = ke^{-at}$

guess $y_p(t)$ find one solution

general solution for (NH): $y_p(t) + ke^{-at}$

now second order case

$$y'' + by' + ky = F(t) \quad \text{(NH)}$$

$$y'' + by' + ky = 0 \quad \text{(H)}$$

we can find general sol of H

$$y_H(t) = k_1 y_1(t) + k_2 y_2(t)$$

guess $y_p(t)$ find one solution

general solution for (NH): $y_H(t) + y_p(t)$

- <u>Example</u>: $y'' + 4y' + 3y = e^{-2t}$

$$y'' + 4y' + 3y = 0$$

$$y_H(t) = k_1 e^{-t} + k_2 e^{-3t}$$

guess: $y_p(t) = Ae^{-2t}$

$$y_p(t) = -e^{-2t}$$

general solution for (NH):

$$y(t) = k_1 e^{-t} + k_2 e^{-3t} - e^{-2t}$$

- <u>Example</u>: $$y'' + y' = e^{-t}$$

$$y_H(t) = k_1 + k_2 e^{-t}$$

guess: $$y_p(t) = Ae^{-t}$$

guess: $$y_p(t) = Ate^{-t}$$

general solution for (NH):

$$y(t) = k_1 + k_2 e^{-t} - te^{-t}$$

Periodic Forcing

$$y'' + by' + ky = A \cdot \cos(\omega t)$$

$A =$ amplitude of forcing

$\frac{2\pi}{\omega} =$ period of forcing

$\frac{\omega}{2\pi} =$ frequency

Damped Case

$$y'' + by' + 2y = \cos(\omega t) \quad b > 0$$

- Example $y'' + 3y' + 2y = \cos(t)$

consider instead

$$y'' + 3y' + 2y = e^{\mathrm{i}t}$$

guess $y_p(t) = Ae^{\mathrm{i}t}$

Use real part of the particular solution

$$y'' + 3y' + 2y = \cos(t)$$

$$k_1 e^{-2t} + k_2 e^{-t} + y_p(t)$$

$$\underbrace{k_1 e^{-2t} + k_2 e^{-t}}_{y_H(t)} + \underbrace{\frac{1}{10}\cos t + \frac{3}{10}\sin t}_{\text{steady state solution } y_p(t)}$$

All solutions approach steady state.

General Case (Damped)

$$y'' + by' + ky = \cos(\omega t) \quad b > 0$$

When damped gen. sol. of (H) approaches 0.

Particular solution of (NH) on form:

$$\alpha \cos(\omega t) + \beta \sin(\omega t)$$

General solution for (NH):

$$\underbrace{\text{GS of H}}_{\to\ 0} + \underbrace{\alpha \cos(\omega t) + \beta \sin(\omega t)}_{\text{steady state}}$$

Undamped Case

$$y'' + by' + ky = \cos(\omega t) \qquad b = 0$$

$$y'' + ky = \cos(\omega t)$$

General solution for (H):

$$k_1 \cos(\sqrt{k} \cdot t) + k_2 \cos(\sqrt{k} \cdot t)$$

Particular solution of (NH) on form:

$$\alpha \cos(\omega t) + \beta \sin(\omega t)$$

General solution for (NH):

$$\underbrace{\text{GS of H}} + \underbrace{\alpha \cos(\omega t) + \beta \sin(\omega t)}_{\text{steady state}}$$