Day 20: August 3rd

- **Chapter: 5.1 Equilibrium Point Analysis**
- **Chapter: 5.2 Qualitative Analysis**
  - Homework:
    - Page 481 #1-11 odd #17-21 odd
    - Rev. Pr: Page 549 #1-5 odd, 9-19 odd, 20-24.
- **Chapter: 6.1 Laplace Transforms**
  - Homework:
    - Page 571 #1, 3, 7, 9, 15, 17.

Recall

$$\star\begin{cases} x' = F(x,y) \\ y' = G(x,y) \end{cases}$$

Non-linear System

Equilibrium point: (x_0, y_0)

evaluated at (x_0, y_0)

Jacobian: $$\mathbf{J}_{|(x_0,y_0)} = \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{pmatrix}$$

near (x_0, y_0) solutions of $\star$ "resemble" solutions of

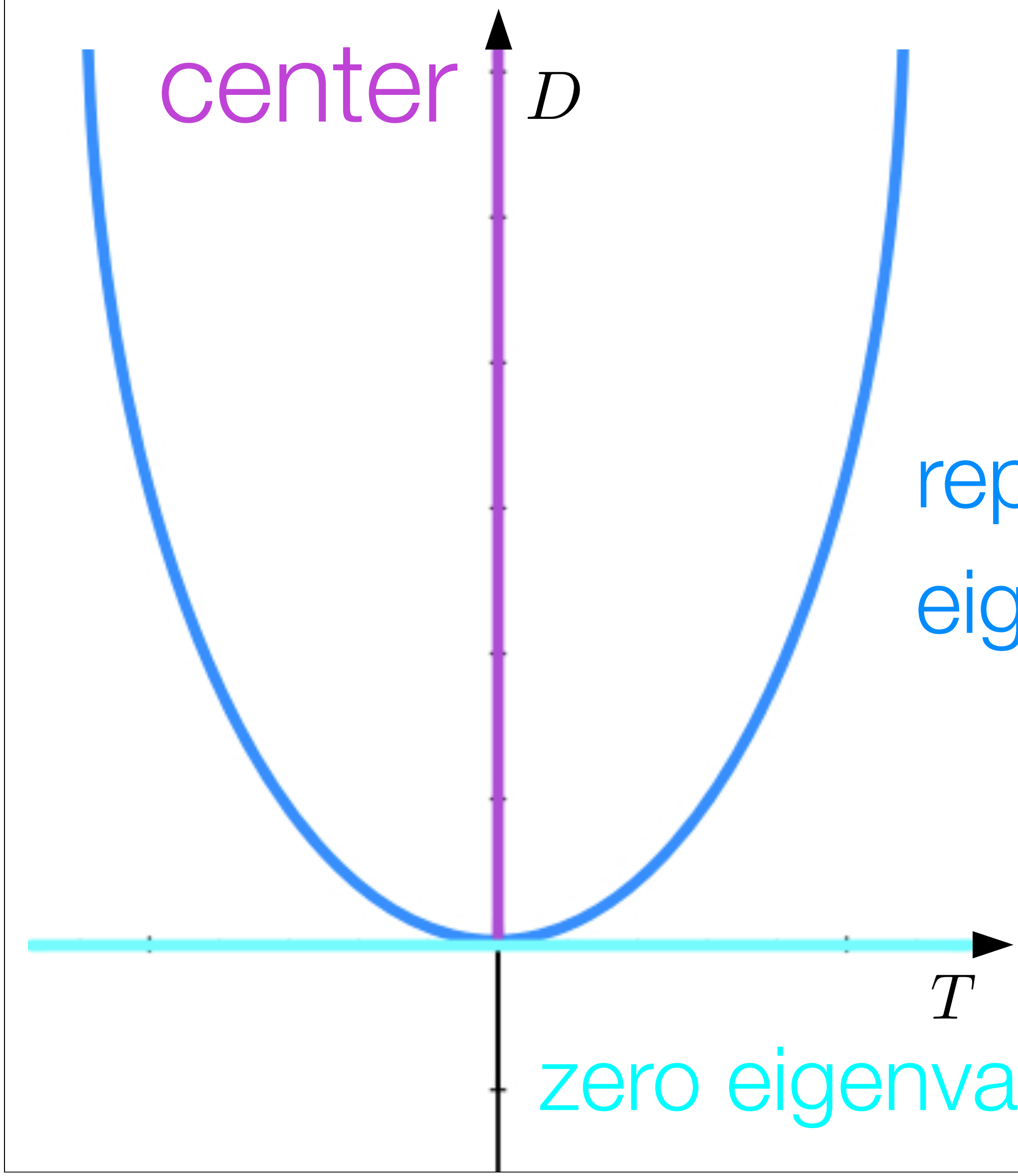
$$\vec{\mathbf{Y}}' = \mathbf{J} \cdot \vec{\mathbf{Y}}$$

near $(0,0)$

Exceptions

- If equilibrium point of linearized system is:
  - center
    - can become spiral source or spiral sink.
  - zero as eigenvalue
    - can become a non-zero eigenvalue.
  - repeated eigenvalue
    - can become non-repeated.

Exceptions



$T^2 - 4D < 0$
$T = 0$

repeated eigenvalue $D = \frac{T^2}{4}$

$D = 0$

- Back to competitive species model

$$\begin{cases} x' = \frac{1}{400}x(400 - x - \frac{3}{2}y) \\ y' = \frac{1}{400}y(400 - y - \frac{3}{2}x) \end{cases}$$

- Equilibrium points: $(0,0), (400,0), (0,400), (160,160)$

- Linearize:

$$\mathbf{J} = \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{pmatrix}$$

$$\mathbf{J} = \begin{pmatrix} 1 - \frac{x}{200} - \frac{3y}{800} & -\frac{3x}{800} \\ -\frac{3y}{800} & 1 - \frac{y}{200} - \frac{3x}{800} \end{pmatrix}$$

- <u>Linearized system at equilibrium points</u>:

$$\mathbf{J}_{|(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- repeated eigenvalue source

$$\mathbf{J}_{|(400,0)} = \begin{pmatrix} -1 & \star \\ 0 & -\frac{1}{2} \end{pmatrix}$$

- sink

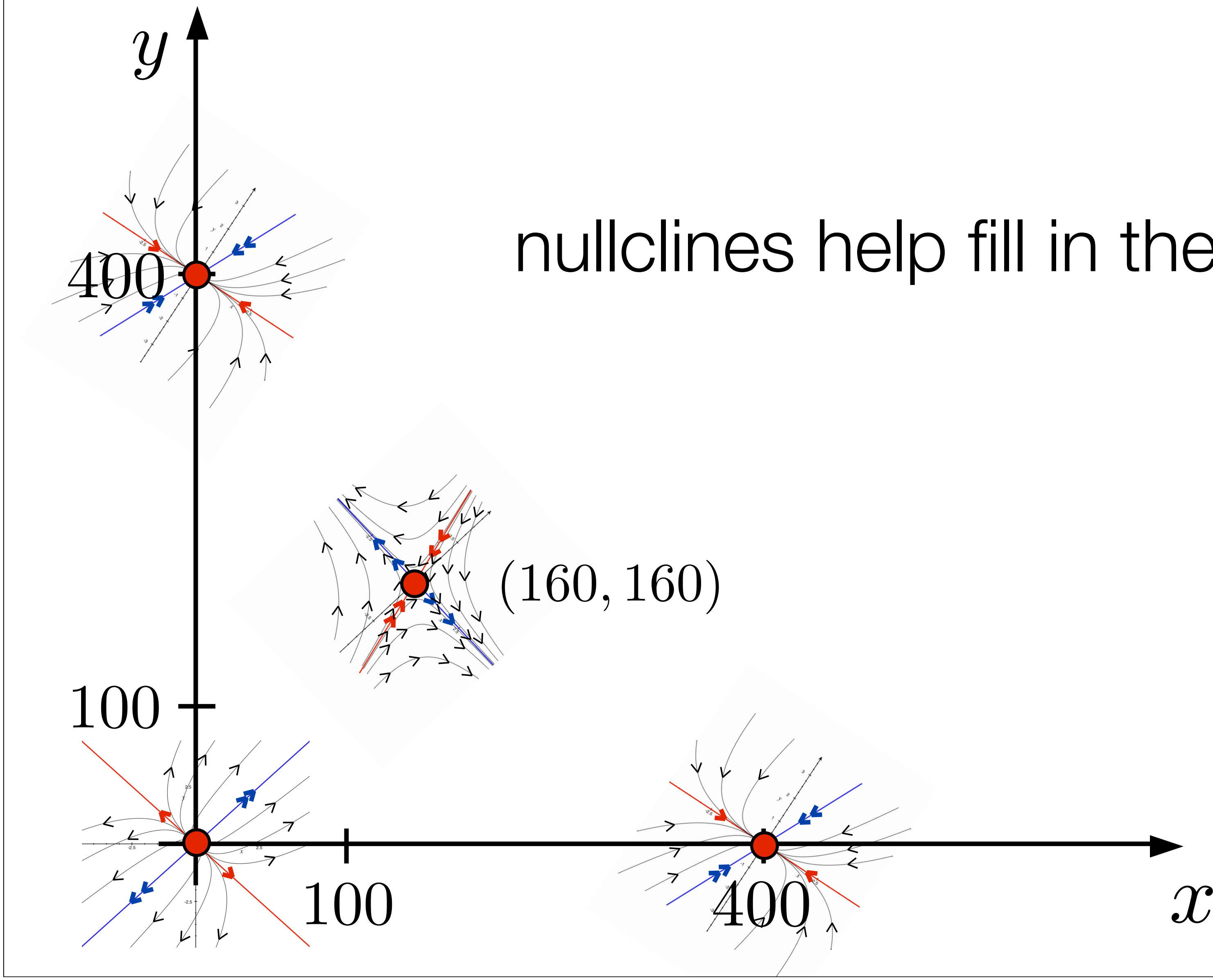
$$\mathbf{J}_{|(0,400)} = \begin{pmatrix} -\frac{1}{2} & \star \\ 0 & -1 \end{pmatrix}$$

- sink

$$\mathbf{J}_{|(160,160)} = \begin{pmatrix} -\frac{2}{5} & -\frac{3}{5} \\ -\frac{3}{5} & -\frac{2}{5} \end{pmatrix}$$

- saddle

- <u>Phase Plane</u>:

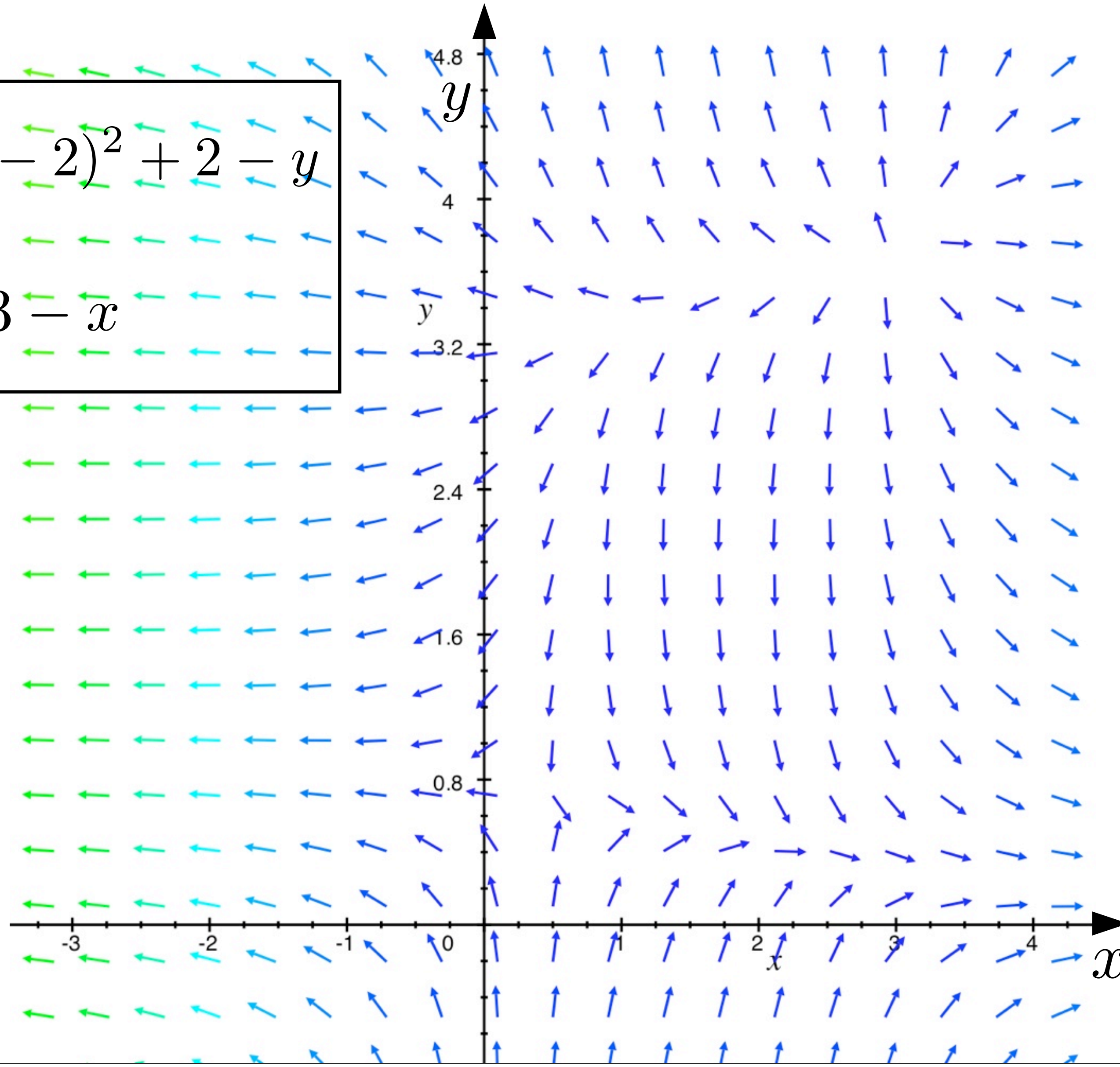


nullclines help fill in the picture

Null-clines

$$\frac{dx}{dt} = (x-2)^3 + (x-2)^2 + 2 - y$$

$$\frac{dy}{dt} = 2(y-2)^2 - 3 - x$$



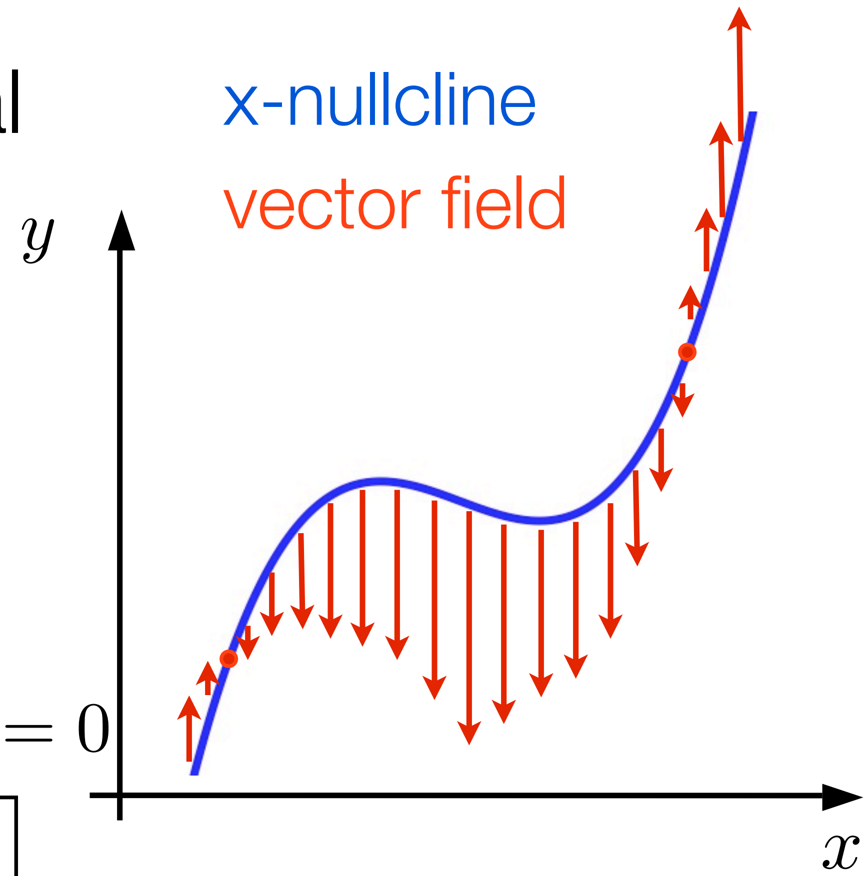
Null-clines

- x-nullcline
  - where $x' = 0$
  - vector field is vertical

$$x' = F(x, y) = 0$$
gives x-nullcline

$$x' = (x-2)^3 + (x-2)^2 + 2 - y = 0$$

$$y = (x-2)^3 + (x-2)^2 + 2$$



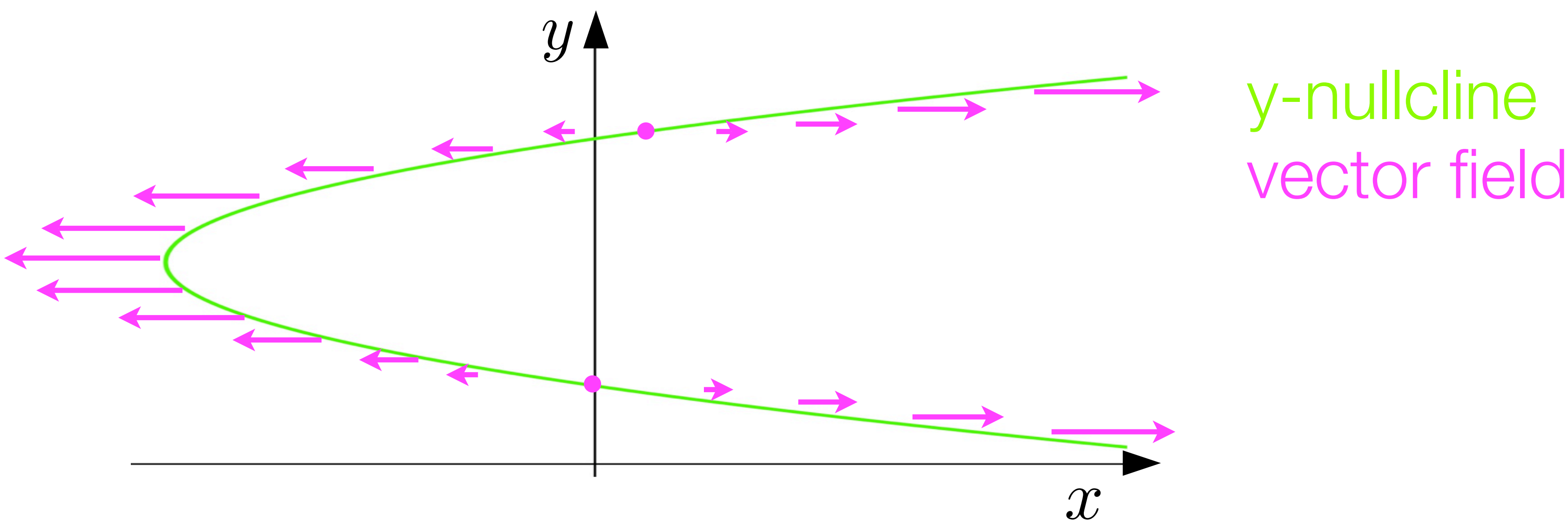
Null-clines

- y-nullcline
  - where $y' = 0$
  - vector field is horizontal

$y' = G(x, y) = 0$
gives y-nullcline

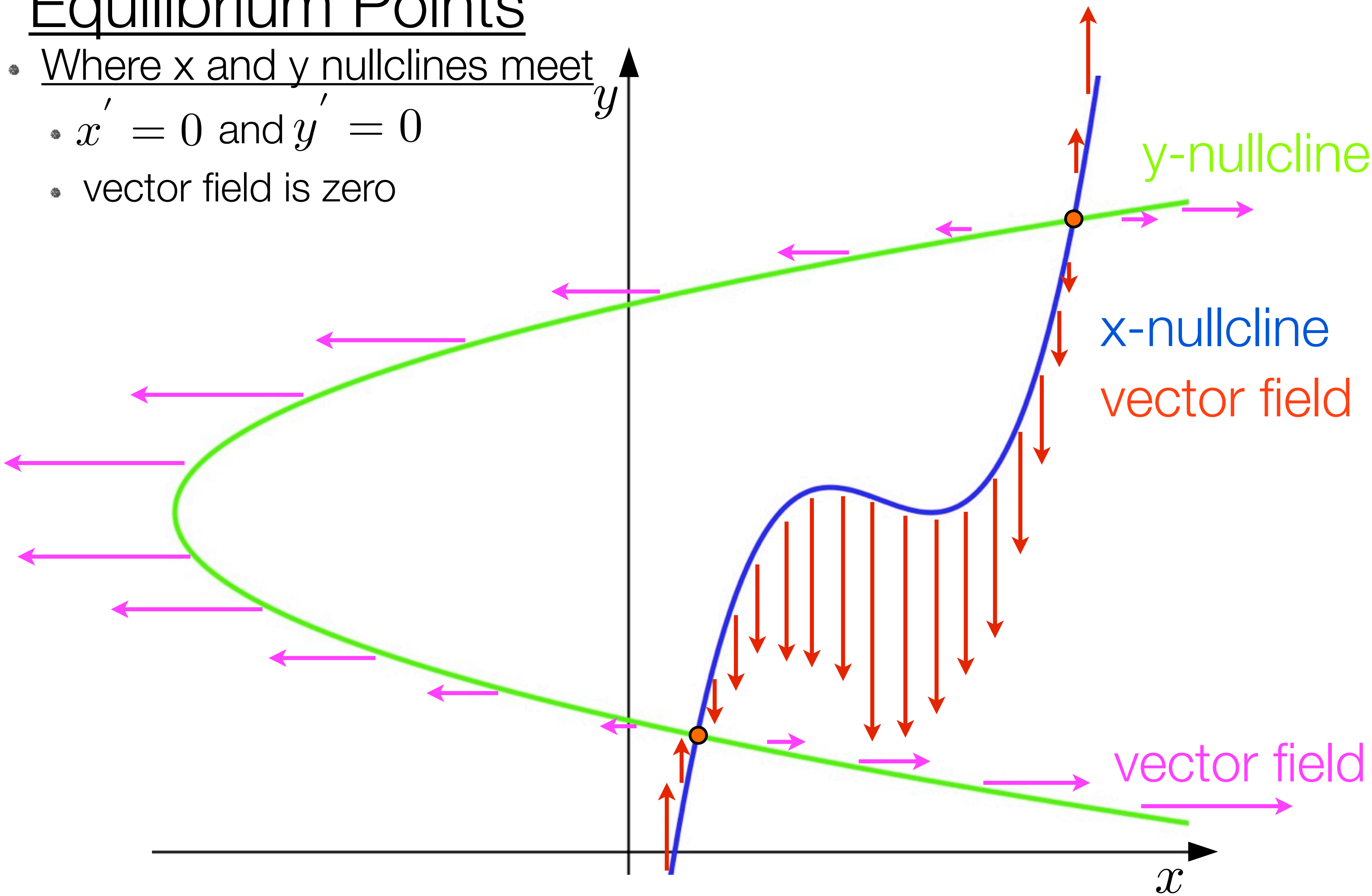
$$y' = 2(y-2)^2 - 3 - x = 0$$

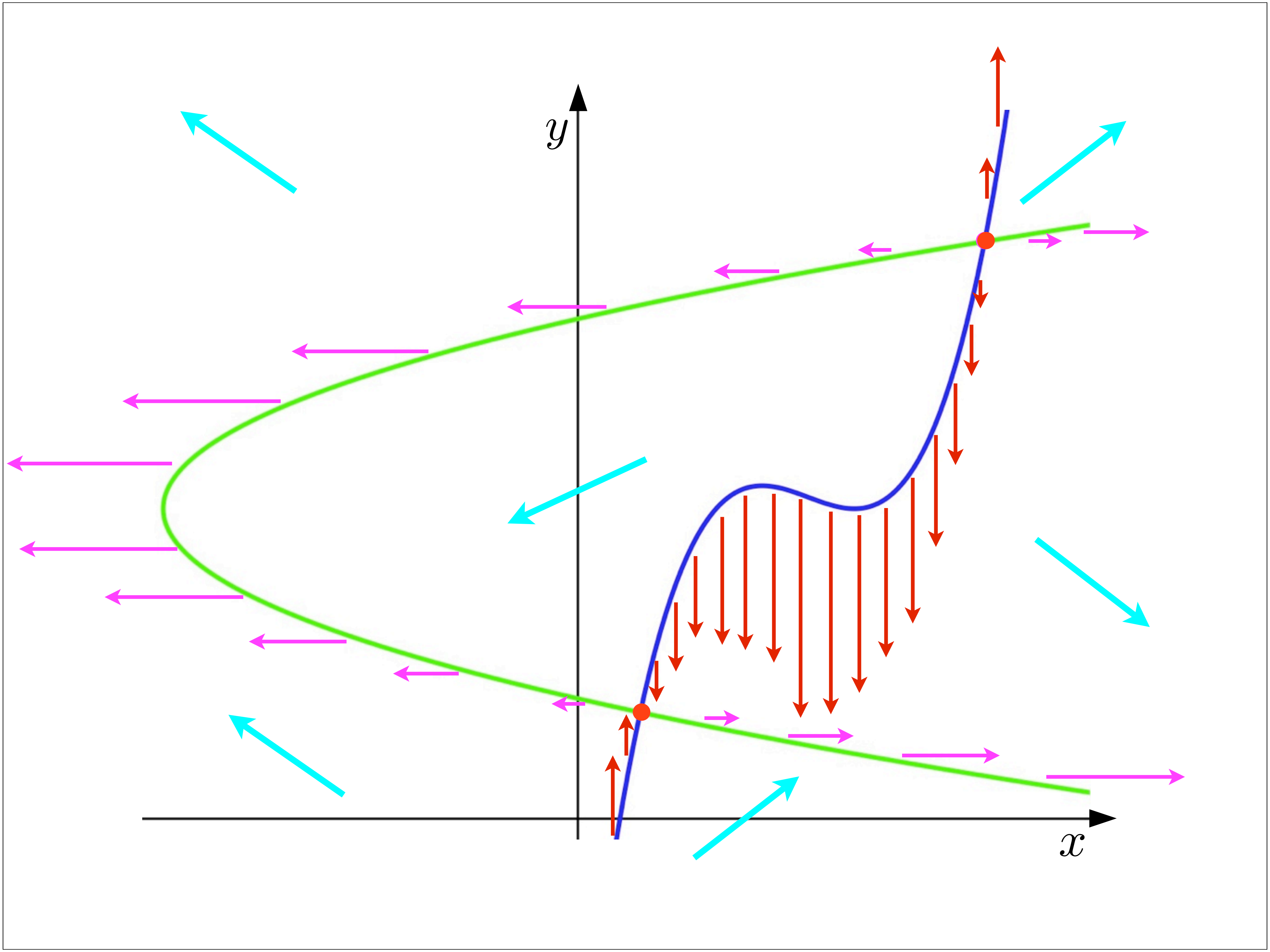
$$x = 2(y-2)^2 - 3$$



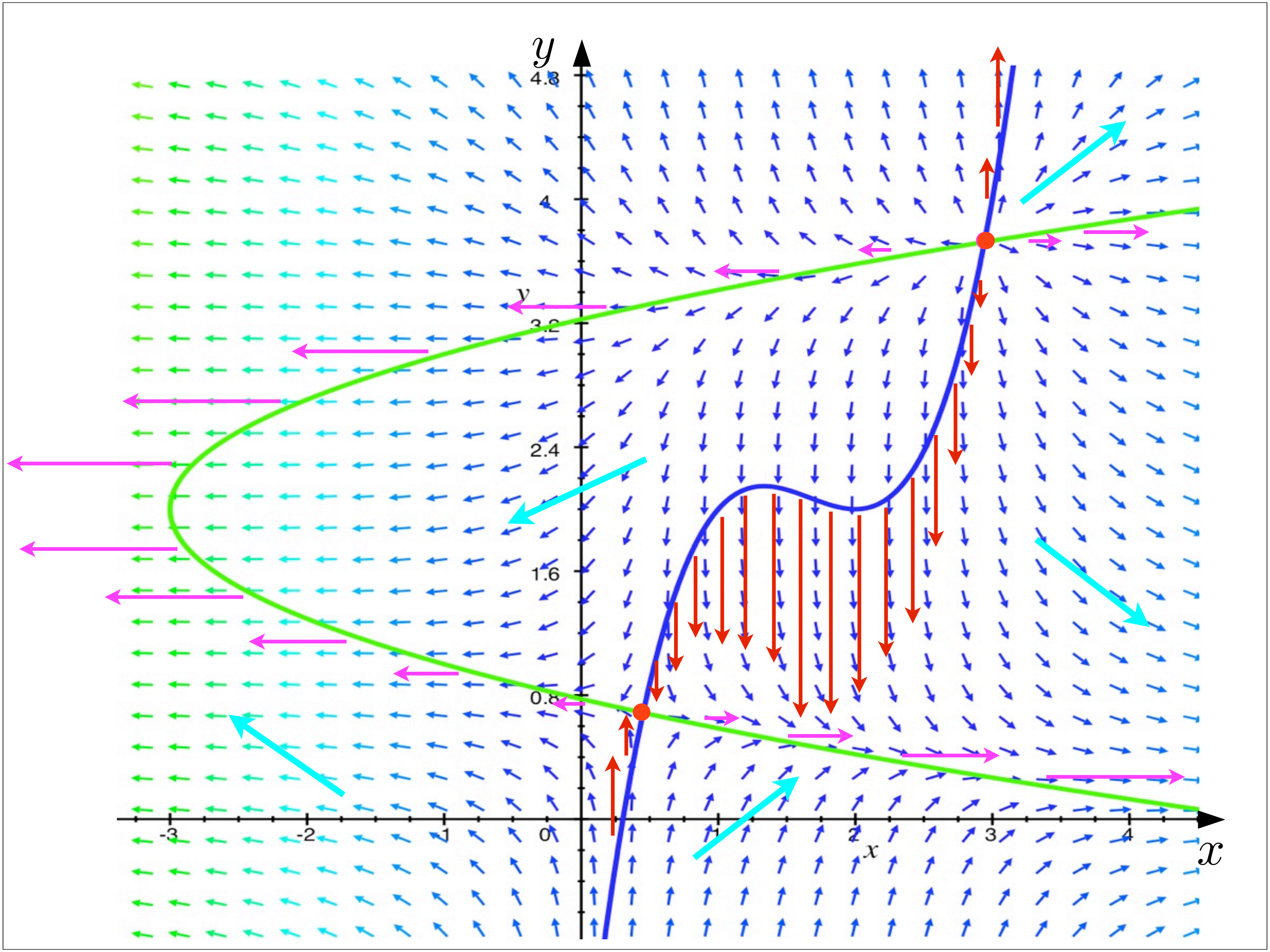
Equilibrium Points

- Where x and y nullclines meet
  - $x' = 0$ and $y' = 0$
  - vector field is zero





y
x



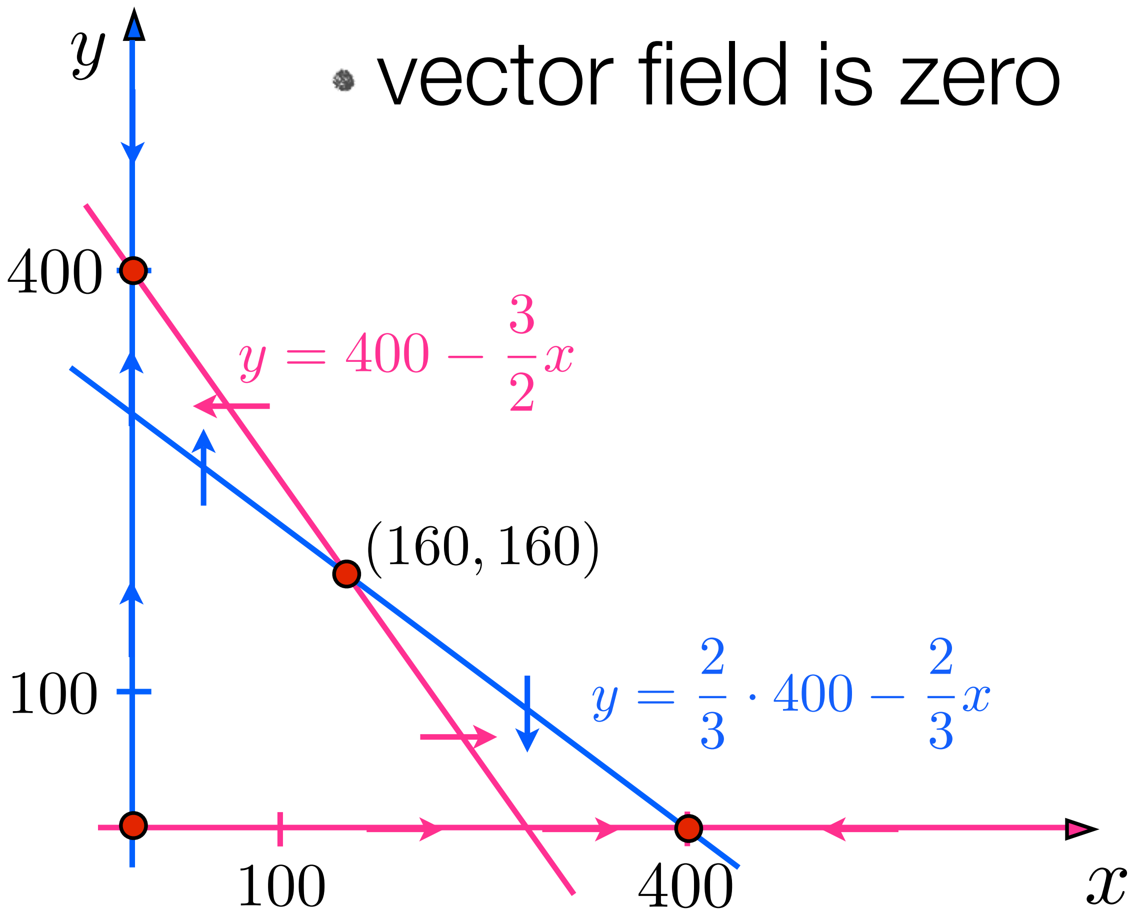
y
4.8
4
3.2
2.4
1.6
0.8
-3
-2
-1
0
1
2
3
4
x

Null-clines

- x-nullcline
  - where $x' = 0$
  - vector field is vertical
- y-nullcline
  - where $y' = 0$
  - vector field is horizontal
- in between nullclines
  - north-east
  - north-west
  - south-east
  - south-west

Equilibrium Points

- Where x and y nullclines meet
  - $x' = 0$ and $y' = 0$
  - vector field is zero



Laplace Transforms

- another way to solve initial value problems

$$y'' + ay' + by = F(t)$$

$$y(0) = \alpha$$
$$y'(0) = \beta$$

especially when forcing is a step function

Laplace Transform

$$y(t) \overset{\mathcal{L}}{\longleftrightarrow} Y(s)$$

t-domain s-domain

Laplace Transforms

$$y(t) \overset{\mathcal{L}}{\longleftrightarrow} Y(s)$$

t-domain | s-domain

- calculus
- diff. eqs.
- integrals

$$\xrightarrow{\mathcal{L}}$$

- algebra
- alg. eqs.
- partial fractions

Laplace Transforms

- Definition

$$\mathcal{L}[y(t)] = Y(s)$$

$$||$$

$$\int_0^\infty e^{-st} y(t) dt$$

integrate with respect
to t and leave s behind

Laplace Transforms

- Example: $\mathcal{L}[c] \quad c \in \mathbb{R}$
- Example: $\mathcal{L}[e^{at}]$

Table

$$\mathcal{L}[c] = \frac{c}{s}, s > 0 \qquad \mathcal{L}^{-1}\left(\frac{c}{s}\right) = c$$

$$\mathcal{L}[e^{at}] = \frac{1}{s-a}, s > a \qquad \mathcal{L}^{-1}\left(\frac{1}{s-a}\right) = e^{at}$$