Day 21: August 4th

- **Chapter: 6 Laplace Transforms**

Chapter 6 not on exam

Review

Laplace Transforms

- another way to solve initial value problems

$$\boxed{y'' + ay' + by = F(t),\ y(0) = \alpha,\ y'(0) = \beta}$$

- linear autonomous systems
  - electrical circuits
  - harmonic oscillators
  - optical devices
  - mechanical systems.

$$t\text{-domain} \quad y(t) \quad \xleftrightarrow{\ \mathcal{L}\ } \quad Y(s) \quad s\text{-domain}$$

Laplace Transform

Laplace Transforms

- Definition: $\mathcal{L}[y(t)] = \int_0^\infty e^{-st} y(t)dt = Y(s)$

- Application: initial value problem

$$\boxed{\begin{array}{c} y' + ay = F(t) \\ y(0) = y_0 \end{array}}$$

$$\boxed{\begin{array}{c} y'' + ay' + by = F(t) \\ y(0) = \alpha, y'(0) = \beta \end{array}}$$

- take Laplace Transform of both sides:

$$\mathcal{L}[y'] + a\mathcal{L}[y] = \mathcal{L}[F(t)]$$

$$\mathcal{L}[y''] + a\mathcal{L}[y'] + b\mathcal{L}[y] = \mathcal{L}[F(t)]$$

- Laplace Transform:

$$\mathcal{L}[y''] = s^2\mathcal{L}[y] - sy(0) - y'(0)$$

$$\mathcal{L}[y'] = s\mathcal{L}[y] - y(0)$$

- Table

$$\mathcal{L}[c] = \frac{c}{s}, s > 0 \qquad \mathcal{L}[e^{at}] = \frac{1}{s-a}, s > a$$

$$\mathcal{L}\left[u_a(t)\right] = \frac{e^{-as}}{s}$$

$$\mathcal{L}[\sin(bt)] = \frac{b}{s^2+b^2} \quad \mathcal{L}[\cos(bt)] = \frac{s}{s^2+b^2}$$

Laplace Transform

diff. eq. solve

t-domain $y(t) \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} Y(s)$ s-domain

Inverse Laplace Transform

Chapter 6 not on exam

Review

- Differential Equations
  - Modeling
    - Come up with models.
    - Analyze models.
  - Solve
    - Analytical: Separate and Integrate
      - Verify Solutions
    - Qualitatively:
    - Numerically: Euler's Method.

First order equations

First Derivative Test

$$\frac{dy}{dt} = F(y)$$

Autonomous equations

Have equilibrium point $F(y_0) = 0$

$F^{'}(y_0) > 0 \Rightarrow$ y_0 is a source

$F^{'}(y_0) < 0 \Rightarrow$ y_0 is a sink

$F^{'}(y_0) = 0 \Rightarrow$ no info about y_0

Non-homogeneous:

$$\boxed{\frac{dy}{dt} + g(t) \cdot y = r(t)}$$

Non separable

1. Find general solution for (H) equation: $y_{\mathrm{H}}(t)$
2. Find particular solution for (NH) equation: $y_{\mathrm{p}}(t)$

- General solution for (NH) equation:

$$y_{\mathrm{NH}}(t) = y_{\mathrm{H}}(t) + y_{\mathrm{p}}(t)$$

Existence and Uniqueness

$$\frac{dy}{dt} = F(y,t)$$

If: Continuously differentiable in y and t, i.e.

F is differentiable in y and t and the derivative is continuous

Then: There exists a unique solution to initial value problem $y(t_0) = y_0$ defined for $t_0 - A < t < t_0 + A$

Result: Solutions can not cross.

Bifurcations

$$\frac{dy}{dt} = F_a(y)$$

A bifurcation is a BIG change in the over all behavior of solutions as a changes

Most common bifurcations

- number of equilibrium point changes
- type of equilibrium point changes

Numerical Method

Euler's Method:

Solve $\dfrac{dy}{dt} = F(y,t) \;\; y(t_0) = y_0$ numerically

Idea: Travel along slope field with step size Δt

Formula

$$y_{n+1} = y_n + F(t_n, y_n) \cdot \Delta t$$

$$t_{n+1} = t_n + \Delta t$$

Linear Systems

$$\vec{\mathbf{Y}}' = \mathbf{A} \cdot \vec{\mathbf{Y}} \text{ where } \vec{\mathbf{Y}} = \begin{pmatrix} x \\ y \end{pmatrix}$$

- **Equilibrium points**
  - If $\det \mathbf{A} \neq 0 \Rightarrow (x, y) = (0, 0)$ only eq point.
  - If $\det \mathbf{A} = 0$ straight line of equilibrium points.
- **Characteristic Equation**

$$\det \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} = 0$$

- Eigenvalues roots of characteristic equation

- **Eigenvectors**

$$\boxed{\mathbf{A} \cdot \vec{\mathbf{v}} = \lambda \vec{\mathbf{v}}}$$

- Finding eigenvectors
  - Non-zero solutions of

$$(\mathbf{A} - \lambda \mathbf{I}) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- always get a redundant system of equations

Real Distinct Eigenvalues

- if we have two distinct real eigenvalues: λ_1, λ_2
  and can find two distinct eigenvectors: $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2$

$$\left.\begin{aligned} \vec{\mathbf{Y}}_1(t) &= e^{\lambda_1 t} \cdot \vec{\mathbf{v}}_1 \\ \vec{\mathbf{Y}}_2(t) &= e^{\lambda_2 t} \cdot \vec{\mathbf{v}}_2 \end{aligned}\right\}$$

two straight line solutions

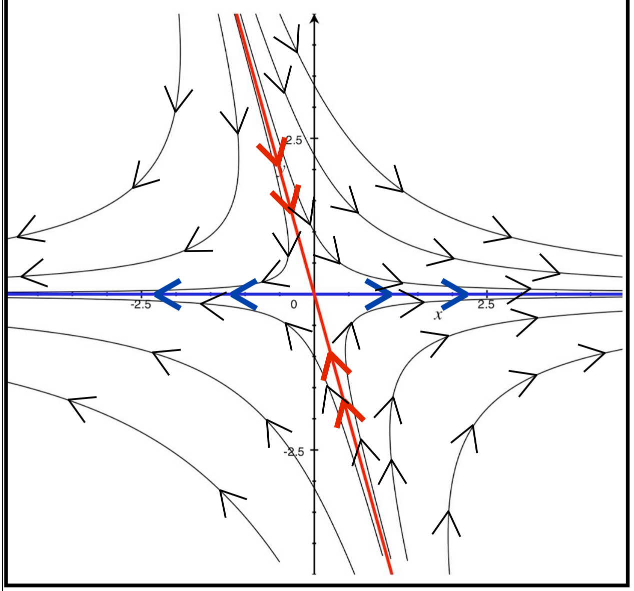
- If $\vec{\mathbf{Y}}_1(0)$ and $\vec{\mathbf{Y}}_2(0)$ are linearly independent

$$\boxed{\begin{aligned} \vec{\mathbf{Y}}(t) &= k_1 \vec{\mathbf{Y}}_1(t) + k_2 \vec{\mathbf{Y}}_2(t) \\ &= k_1 e^{\lambda_1 t} \cdot \vec{\mathbf{v}}_1 + k_2 e^{\lambda_2 t} \cdot \vec{\mathbf{v}}_2 \end{aligned}}$$

Real Distinct Eigenvalues

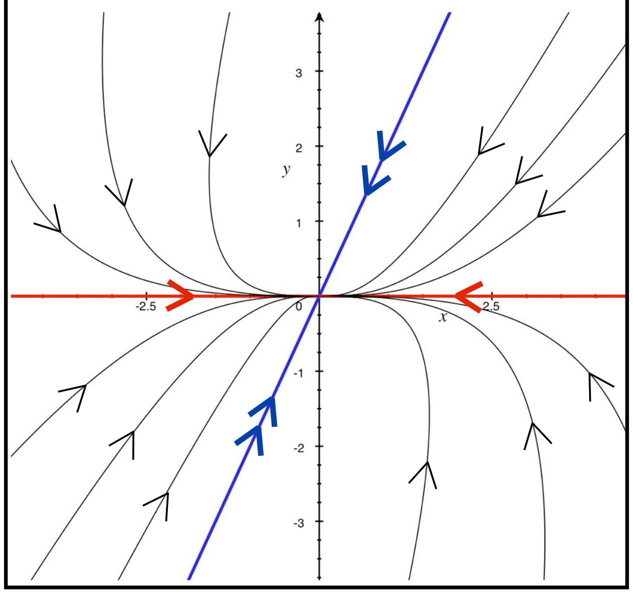
Saddle

$\lambda_1 < 0 < \lambda_2$



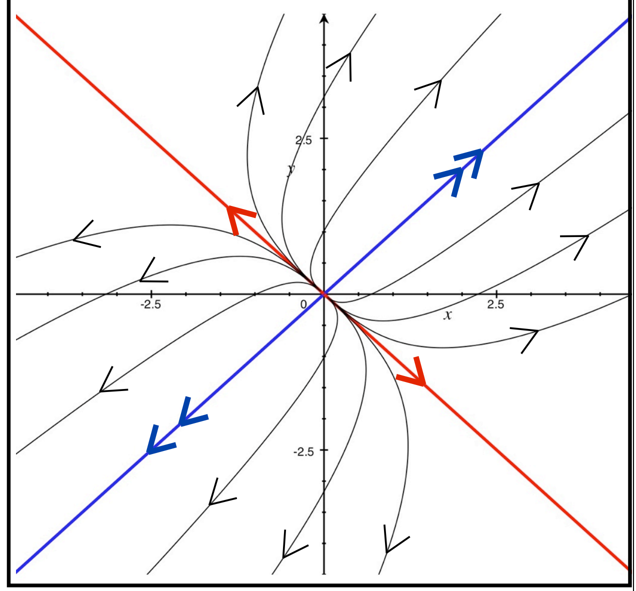
Sink

$\lambda_2 < \lambda_1 < 0$



Source

$0 < \lambda_1 < \lambda_2$



$$\lambda_1, \lambda_2 \in \mathbb{R}$$

Complex Eigenvalues

- One solution:

$$\vec{\mathbf{Y}}_1(t) = e^{\lambda_1 t} \cdot \vec{\mathbf{v}}_1$$

break up solution

$\vec{\mathbf{Y}}_{\mathrm{Re}}(t)$ — Real part of solution

$\vec{\mathbf{Y}}_{\mathrm{Im}}(t)$ — Imaginary part of solution

- Fact: Both $\vec{\mathbf{Y}}_{\mathrm{Re}}(t)$ and $\vec{\mathbf{Y}}_{\mathrm{Im}}(t)$ are solutions.
- If $\vec{\mathbf{Y}}_{\mathrm{Re}}(t)$ and $\vec{\mathbf{Y}}_{\mathrm{Im}}(t)$ are linearly independent

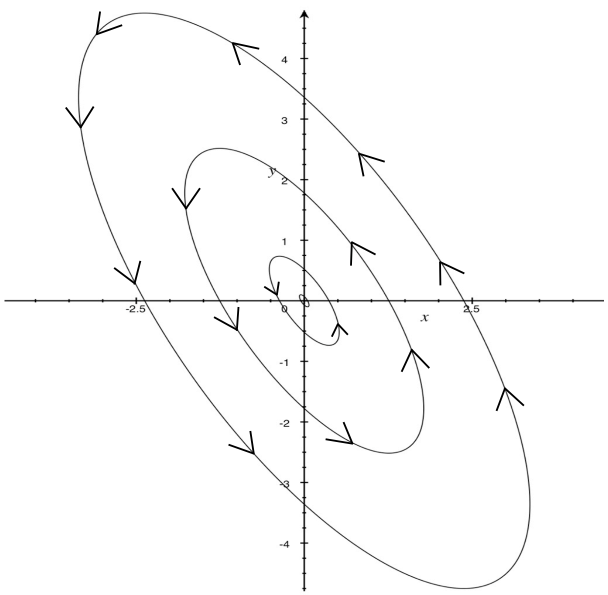
Gen sol: $\vec{\mathbf{Y}}(t) = k_1 \vec{\mathbf{Y}}_{\mathrm{Re}}(t) + k_2 \vec{\mathbf{Y}}_{\mathrm{Im}}(t)$

Complex Eigenvalues

$$\lambda = a \pm \mathrm{i}b$$

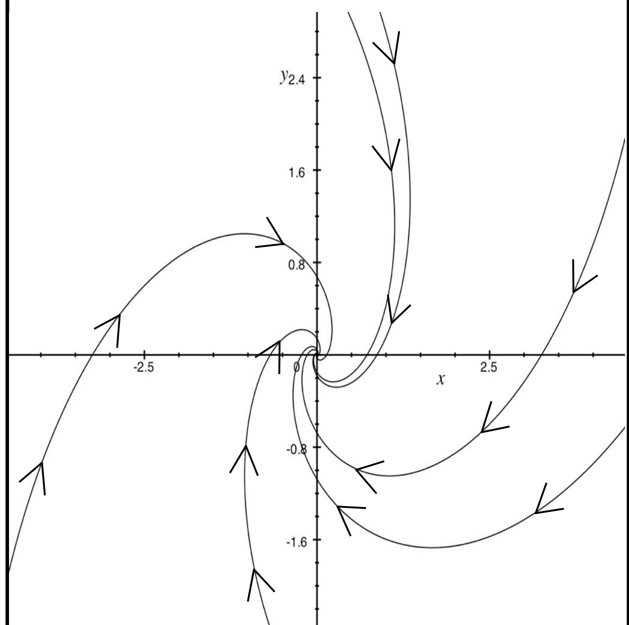
Center

$$a = 0$$



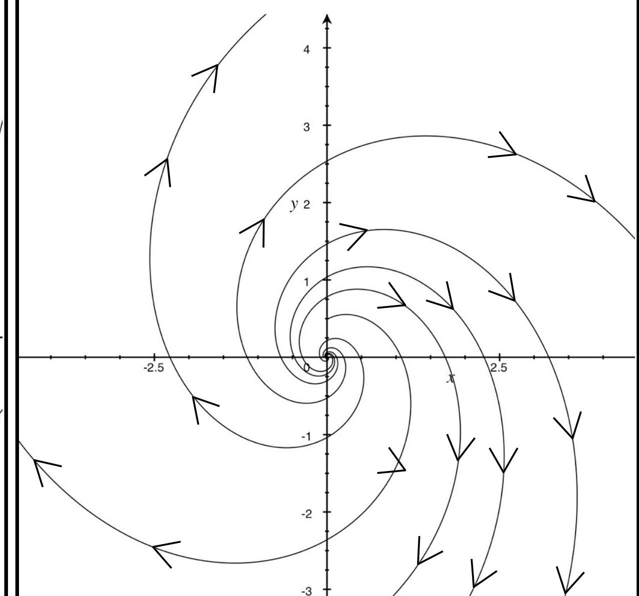
Spiral Sink

$$a < 0$$



Spiral Source

$$a > 0$$

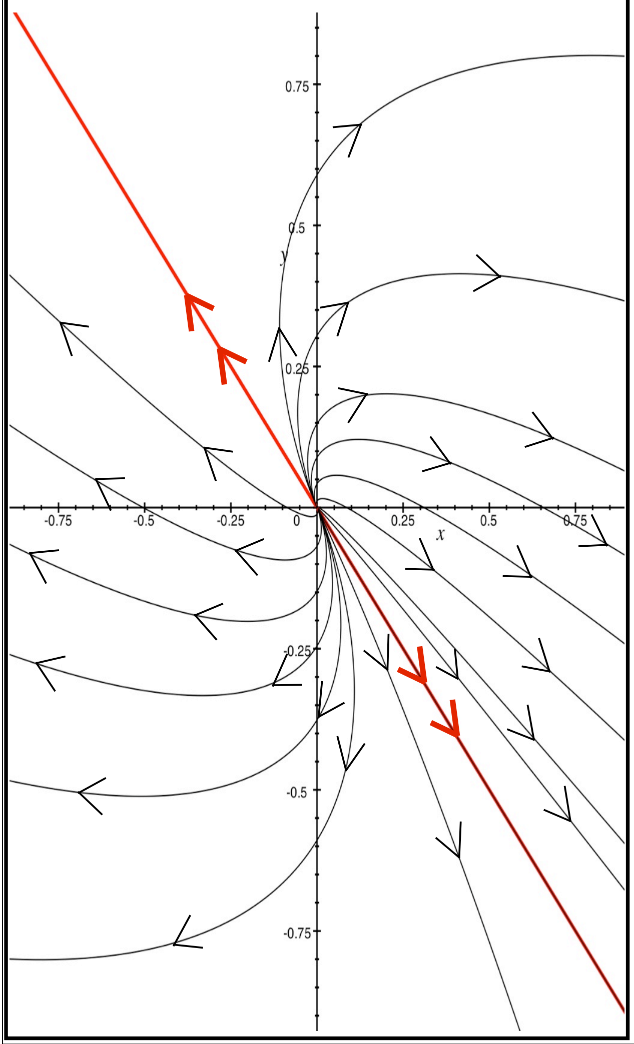


In all cases rotation can be clockwise or counter clockwise.

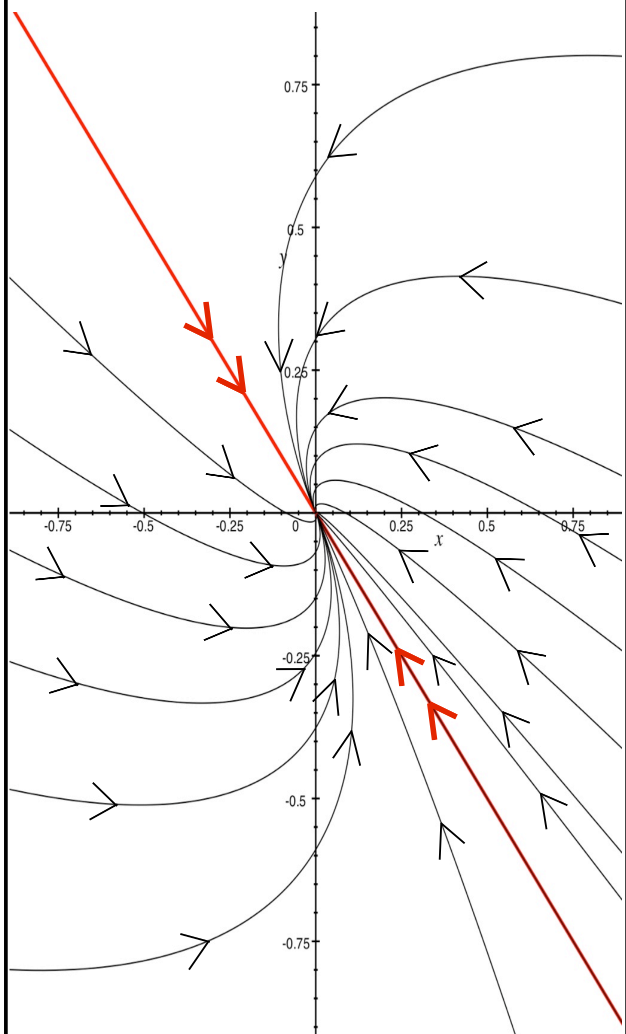
Review: Real Repeated Eigenvalues

$$y(t) = k_1 e^{\lambda t} v_1 + k_2 (t e^{\lambda t} v_1 + e^{\lambda t} v_2)$$

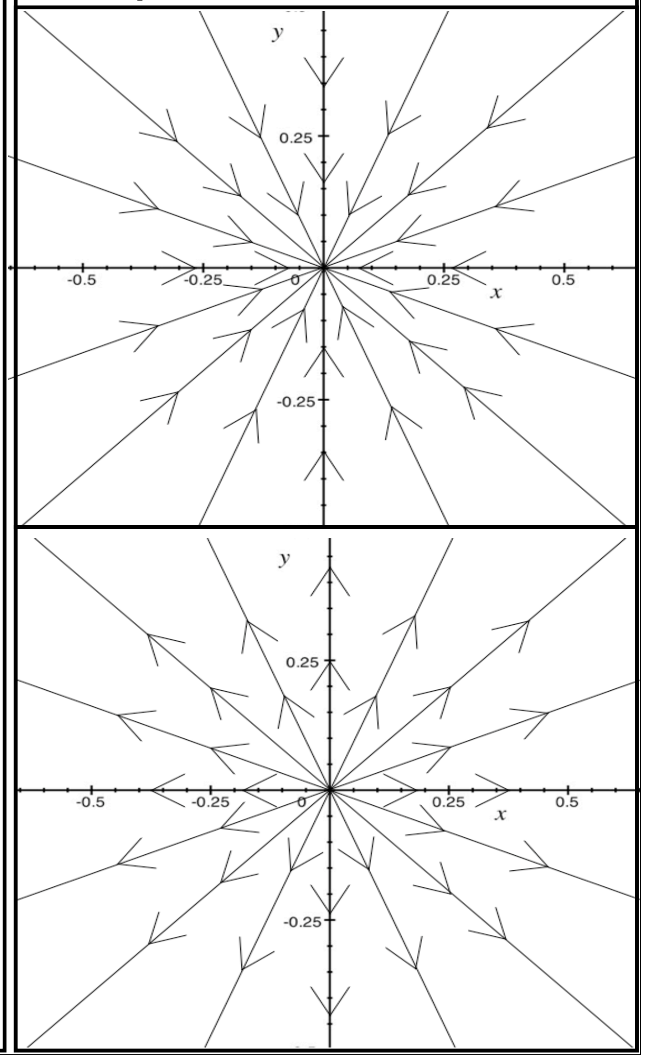
$\lambda > 0$



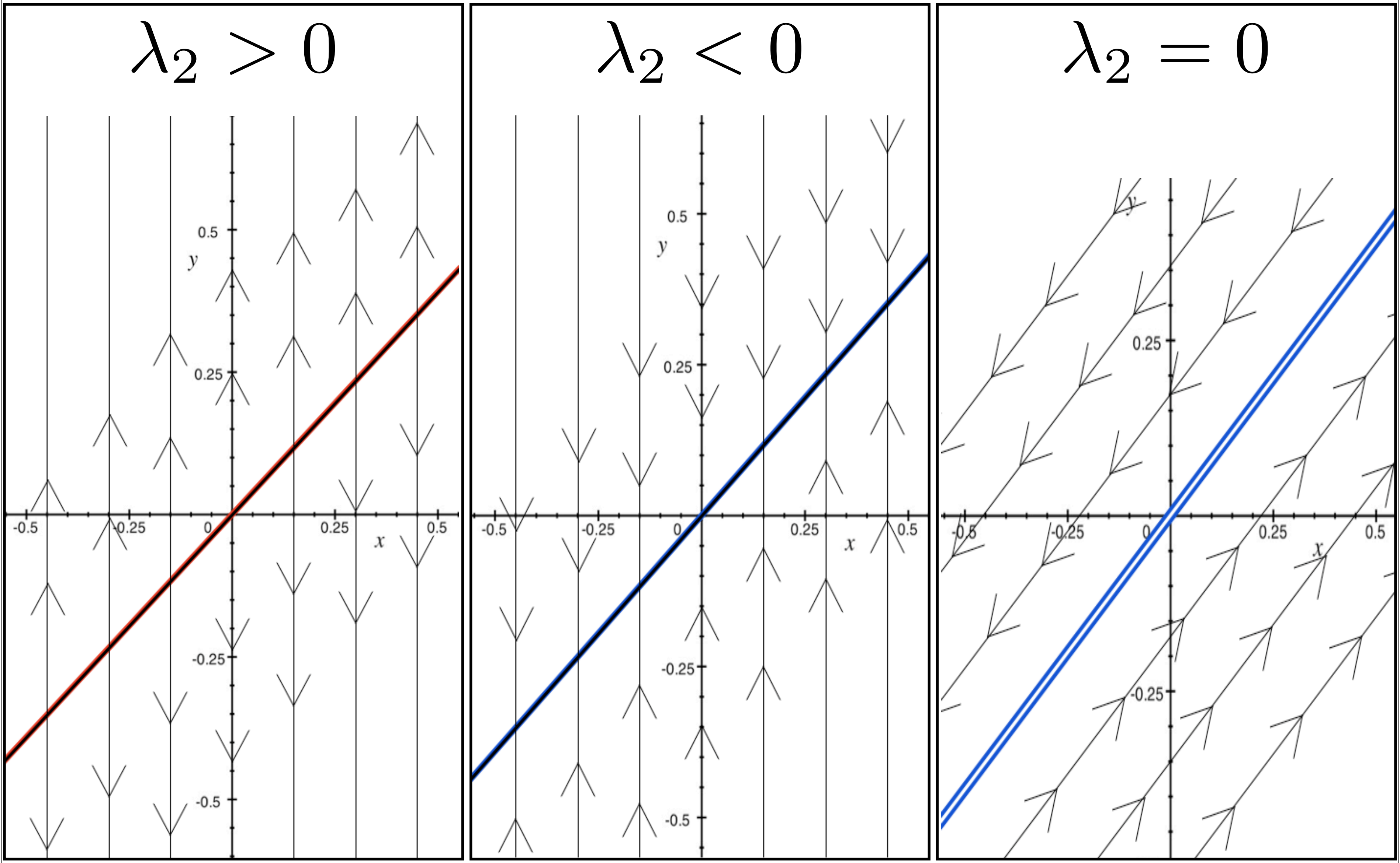
$\lambda < 0$

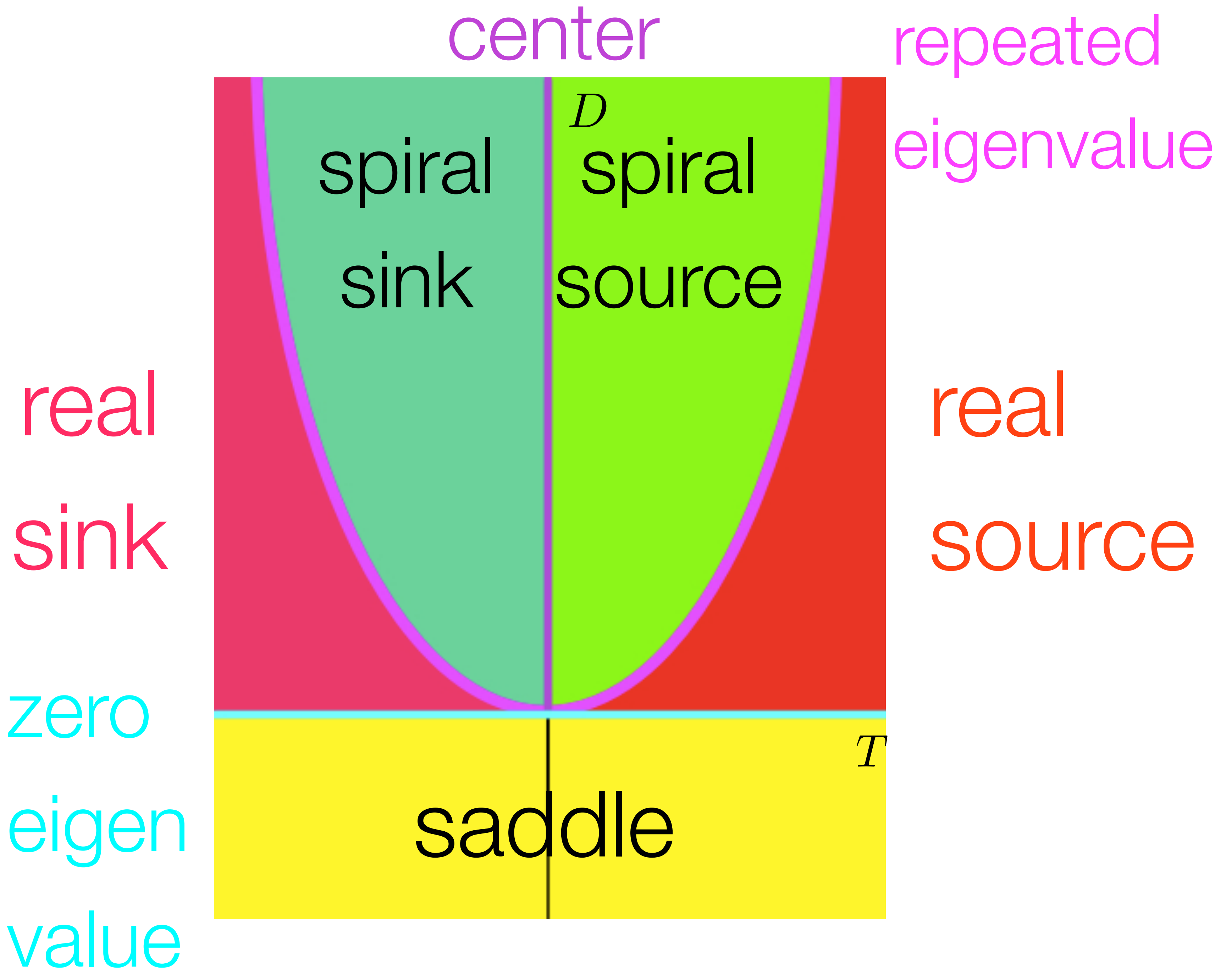


special case



Review: Zero Eigenvalue




center
repeated
eigenvalue
D
spiral
sink
spiral
source
real
sink
real
source
zero
eigen
value
T
saddle

- Strategy for solving: $y'' + by' + ky = 0$

  guess: $y = e^{st}$

  $$s^2 + bs + k = 0$$

  find roots: s_1, s_2

- Case 1

  $s_1, s_2 \in \mathbb{R} \quad s_1 \neq s_2$

  get solutions: $y_1(t) = e^{s_1 t} \quad y_2(t) = e^{s_2 t}$

  general solution: $y(t) = k_1 e^{s_1 t} + k_2 e^{s_2 t}$

- Case 2: complex roots: $s = \alpha \pm \mathrm{i}\beta$
  - Formula for general solution:

$$y(t) = e^{\alpha t}(k_1 \cos(\beta t) + k_2 \sin(\beta t))$$

- Case 3: repeated roots: s_1

get solutions: $y_1(t) = e^{s_1 t} \quad y_2(t) = t \cdot e^{s_1 t}$

general solution: $y(t) = k_1 e^{s_1 t} + k_2 t e^{s_1 t}$

Non-linear Systems

$$\frac{dx}{dt} = F(x, y)$$

$$\frac{dy}{dt} = G(x, y)$$

- Two techniques:
  - Equilibrium Point Analysis
    - Linearization
  - Nullclines

Equilibrium Point Analysis

$$\star \begin{cases} x' = F(x,y) \\ y' = G(x,y) \end{cases}$$

Non-linear System

Equilibrium point: (x_0, y_0)

evaluated at (x_0, y_0)

Jacobian: $$\mathbf{J}_{|(x_0,y_0)} = \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{pmatrix}$$

near (x_0, y_0) solutions of $\star$ "resemble" solutions of

$$\vec{\mathbf{Y}}' = \mathbf{J} \cdot \vec{\mathbf{Y}}$$

near $(0, 0)$

Null-clines

- x-nullcline
  - where $x' = 0$
  - vector field is vertical

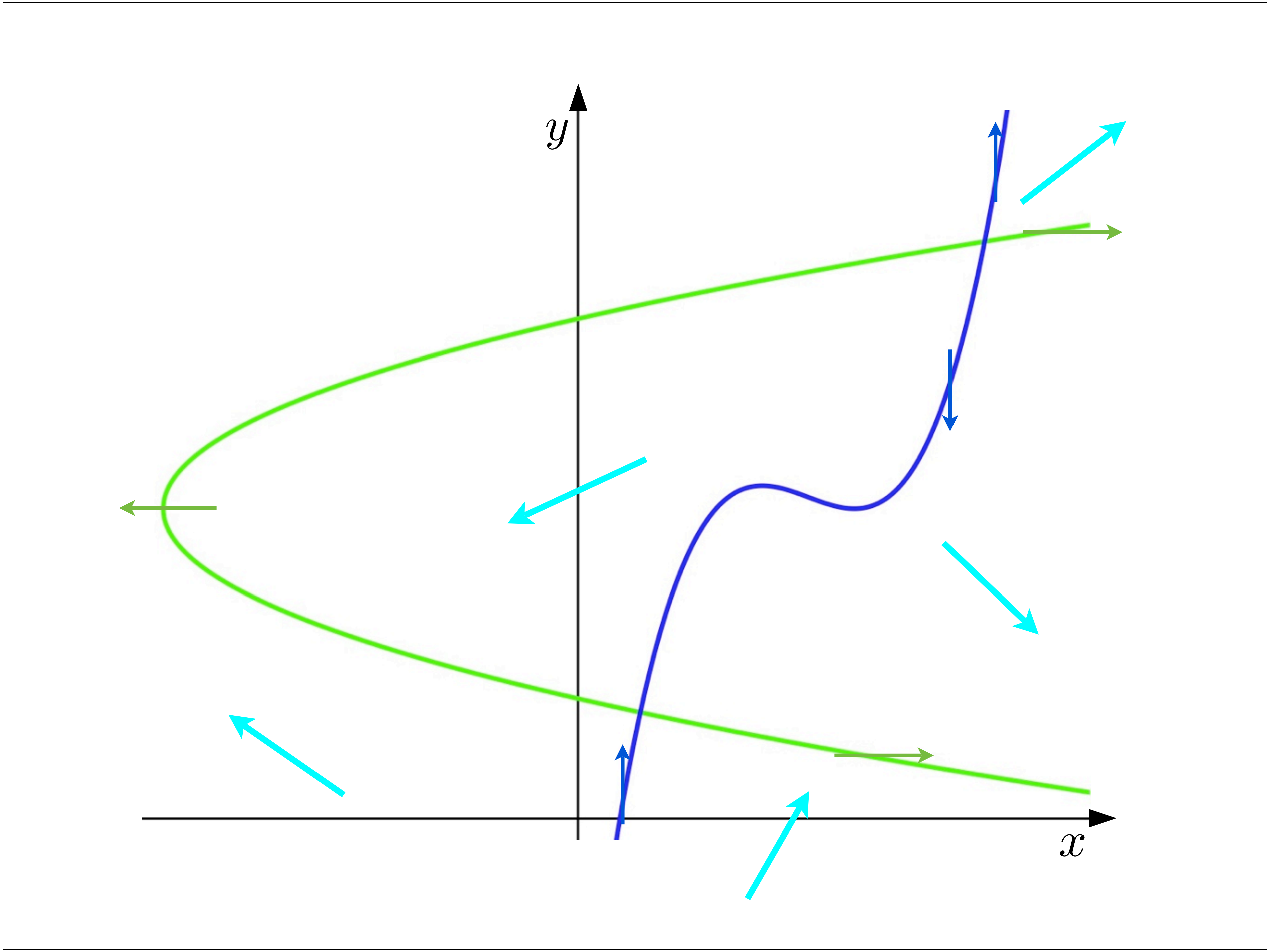
$$x' = F(x, y) = 0$$

gives x-nullcline

- y-nullcline
  - where $y' = 0$
  - vector field is horizontal

$$y' = G(x, y) = 0$$

gives y-nullcline



y
x