Day 9: July 12th

- **Chapter: 2.1 Modeling via Systems**
- Homework:
    - 1.1 Page 19 #19, 20.
    - 2.1 Page 164 #1-4, 9, 10, 15.

- **Chapter: 2.2 The Geometry of Systems**
- Homework:
    - Page 182 #1, 3, 9, 11-27 odd.

Part II: Systems of ODEs

- **Things in the real world are interconnected**
  - Populations:
    - #predators ⟺ #prey
  - Weather:
    - temperature ⟺ wind-speeds
  - Financial markets:
    - unemployment ⟺ spending
  - Neuroscience:
    - hippocampus ⟺ entorhinal cortex
- **WARNING things can get complicated**

- **Chapter: 2.1 Modeling via Systems**
  - Unlimited growth

$$\frac{dP}{dt} = kP$$

just depends on the population

  - Logistic population model

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right)$$

  - Interacting Populations:

    Predator-Prey System:
      - rabbits.
      - foxes.

Predator-Prey Systems

- Assumptions
- Rabbits:
  - reproduce at a rate prop. to population if no foxes.
  - foxes eat rabbits at a rate prop. to # of rabbit/fox meetings.
- Foxes
  - if no rabbits, fox population decreases at a rate proportional to population
  - fox population increases at a rate prop. to # of rabbit/fox meetings.

Predator-Prey Systems

- Variables

  $t =$ time

  $R = R(t) =$ # rabbits

  $F = F(t) =$ # foxes

- Describe change with time

$$\frac{dR}{dt} = ?$$

$$\frac{dF}{dt} = ?$$

$$\longrightarrow \quad R(t), \quad F(t)$$

Predator-Prey Systems

- Model Rabbits
  - reproduce at a rate prop. to population if no foxes.

    $$\frac{dR}{dt} = k \cdot R \text{ if only rabbits}$$

  - foxes eat rabbits at a rate prop. to # of rabbit/fox meetings: $\alpha \cdot R \cdot F$

    $$\frac{dR}{dt} = k \cdot R - \alpha \cdot R \cdot F$$

Predator-Prey Systems

- Model Foxes
  - if no rabbits, fox population decreases at a rate proportional to population

$$\frac{dF}{dt} = -l \cdot F \text{ if no rabbits}$$

  - fox population increases at a rate prop. to # of rabbit/fox meetings

$$\frac{dF}{dt} = -l \cdot F + \beta \cdot R \cdot F$$

Predator-Prey Systems

- System of equations:

$$\frac{dR}{dt} = k \cdot R - \alpha \cdot R \cdot F$$

$$\frac{dF}{dt} = -l \cdot F + \beta \cdot R \cdot F$$

  Two differential equations

- Non-linear.
- Must solve both at the same time.

Predator-Prey Systems

- More general case:

$$\frac{dR}{dt} = k \cdot R \cdot \left(1 - \frac{R}{N}\right) - \alpha \cdot R \cdot F$$

$$\frac{dF}{dt} = -l \cdot F + \beta \cdot R \cdot F$$

Two differential equations

- Non-linear.
- Must solve both at the same time.

What can we do?

- Geometry of systems
  - Section 2.2
- Look at a simpler case: linear systems
  - Chapters 3 and 4
- Return to non-linear systems.
  - linearize
  - develop new tools
  - Chapter 5

Predator-Prey Systems

- Special case:

$$\frac{dR}{dt} = R \cdot \Big(1 - F\Big)$$ rabbits

$$\frac{dF}{dt} = F \cdot \Big(-1 + R\Big)$$ foxes

- First: Equilibrium Points.

$$\frac{dR}{dt} = 0 \text{ and } \frac{dF}{dt} = 0$$

Predator-Prey Systems

- Special case:

$$\frac{dR}{dt} = R \cdot \Big(1 - F\Big)$$ rabbits

$$\frac{dF}{dt} = F \cdot \Big(-1 + R\Big)$$ foxes

if no foxes: $F = 0$

$$\frac{dF}{dt} = 0 \qquad \frac{dR}{dt} = R$$

$$F(t) = 0 \qquad R(t) = c \cdot e^t$$

Predator-Prey Systems

- Special case:

$$\frac{dR}{dt} = R \cdot \Big(1 - F\Big)$$ rabbits

$$\frac{dF}{dt} = F \cdot \Big(-1 + R\Big)$$ foxes

if no rabbits: $R = 0$

$$\frac{dR}{dt} = 0 \qquad \frac{dF}{dt} = -F$$

$$R(t) = 0 \qquad F(t) = \bar{c} \cdot e^{-t}$$

Predator-Prey Systems

- Special case:

$$\frac{dR}{dt} = R \cdot \Big(1 - F\Big) \qquad \text{rabbits}$$

$$\frac{dF}{dt} = F \cdot \Big(-1 + R\Big) \qquad \text{foxes}$$

- Phase Plane: 2-D equivalent to phase line
  - solutions travel in phase plane.
  - PredatorPrey software.

Predator-Prey Systems

- Vector field: equivalent to slope field
  - plot slope in phase plane (F vs R)

    at (R, F) plot $V(R,F) = \left(\frac{dR}{dt}, \frac{dF}{dt}\right)$
  - a solution curve is always tangent to V.
    - solutions follow V.
  - only for autonomous equations.
- Direction field = scaled vector field
  - all vectors set to the same length.