INTRODUCTION

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We review the cohomology of algebraic varieties, leading to a question of how to detect good reduction of elliptic curves at \( p \) inside the \( p \)-adic étale cohomology. All errors are mine. Comments and corrections welcome. For missing references, one might check the syllabus located at the seminar website http://math.bu.edu/people/bergdall/seminars/padic-hodge-seminar.html.

1. Cohomology

Suppose that \( X/\mathbb{Q} \) is an algebraic variety, proper\(^1\) and smooth. There are various linearizing operations we can perform on \( X \).

1.1. Singular cohomology. Look at the complex manifold \( X(\mathbb{C}) \), just as a topological space. It has singular cohomology groups \( H^i_{\text{sing}}(X(\mathbb{C}), \mathbb{Q}) \). These are finite-dimensional \( \mathbb{Q} \)-vector spaces, concentrated in degrees \( 0 \leq i \leq 2 \dim X \). One invariant(s) of \( X \) is given by the Betti numbers

\[
b_i(X) := \dim_{\mathbb{Q}} H^i_{\text{sing}}(X(\mathbb{C}), \mathbb{Q}).
\]

Since an elliptic curve is topologically a torus, \( b_0(E) = b_2(E) = 1 \) and \( b_1(E) = 2 \).

1.2. de Rham cohomology. One could also take the algebraic de Rham cohomology \( H^i_{\text{dR}}(X/\mathbb{Q}) \), see [5]. Again these are finite-dimensional \( \mathbb{Q} \)-vector spaces but the definition of de Rham cohomology (as the hypercohomology of the de Rham complex \( \Omega^\bullet_{X/\mathbb{Q}} \)) gives a spectral sequence

\[
E_1^{p,q} = H^p(X, \Omega^q_{X/\mathbb{Q}}) \Rightarrow H^{p+q}_{\text{dR}}(X/\mathbb{Q}).
\]

The famous degeneration of this sequence (Hodge theory over \( \mathbb{C} \), a proof due to Faltings over \( \mathbb{Q}_p \) or a proof by Deligne and Illusie which reduces the situation to characteristic \( p \) [3]) implies that for each \( i \), there is a filtration on \( H^i_{\text{dR}}(X/\mathbb{Q}) \) whose associated graded recovers Hodge cohomology spaces

\[
Gr^* H^i_{\text{dR}}(X/\mathbb{Q}) = \bigoplus_{p+q=i} H^p(X, \Omega^q_{X/\mathbb{Q}}).
\]

In particular, forgetting the filtration, \( H^i_{\text{dR}}(X/\mathbb{Q}) \) is the right hand side as \( \mathbb{Q} \)-vector spaces. If \( E \) is an elliptic curve then the interesting degree is \( i = 1 \) and

\[
H^0(E, \Omega^1_{X/\mathbb{Q}}) = \mathbb{Q} \cdot \omega
\]

is generated by an invariant differential and \( H^1(E, \mathcal{O}) \) is Serre dual to \( H^0(E, \Omega^1_{X/\mathbb{Q}}) \). Thus \( \dim_{\mathbb{Q}} H^1_{\text{dR}}(X/\mathbb{Q}) = 2 \), as expected.

Moreover, there are comparisons with singular cohomology after tensoring with a period ring. To explain this, consider an \( i \)-cycle \( \gamma \in H_i(X(\mathbb{C}), \mathbb{Q}) \) and an \( i \) form \( \omega \in H^1(X/\mathbb{Q}) \). We can form the number

\[
\langle \gamma, \omega \rangle := \int_\gamma \omega.
\]

But simple\(^2\) examples reveal that this is probably not rational (or algebraic even). But this pairing does define an isomorphism

\[
H^i(X(\mathbb{C}), \mathbb{C}) = H^i_{\text{sing}}(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \simeq H^i_{\text{dR}}(X/\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} = H^i_{\text{dR}}(X_{\mathbb{C}}/\mathbb{C})
\]

\(^1\)Assume its projective if you like, or even a projective curve, or that its an elliptic curve!

\(^2\)\(\int_{|z|=1} \frac{dz}{z} = 2\pi i\)
Note that the left and right hand sides are defined completely differently, even if after a course in algebraic topology they become synonymous. Combining with the Hodge-de Rham degeneration this gives “Hodge theory”. We want to emphasize this is also a prototype for a comparison theorem: one can compare cohomology theories after extending scalars to a certain period ring.

1.3. Étale cohomology. The final cohomology theory we’re going to discuss is the $p$-adic étale cohomology. So choose a prime $p$. Then Grothendieck, Deligne, etc. have defined an studied cohomology groups $H^i_\text{ét}(X_{\overline{Q}}, \mathbb{Q}_p)$. There are finite-dimensional $\mathbb{Q}_p$-vector spaces whose degrees match the Betti numbers. Moreover, each space has symmetries by the Galois group $G_{\mathbb{Q}} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (which acts via functoriality from the maps $X_{\overline{Q}} \to X_{\overline{Q}}$).

To orient yourself, consider an elliptic curve $E$. Then it’s known that $E[p^n](\overline{Q}) \cong (\mathbb{Z}/p^n\mathbb{Z})^ {\oplus 2}$, and moreover

(a) multiplication by $p$ induces natural maps $E[p^n]|(\overline{Q}) \to E[p^{n-1}](\overline{Q})$ (so that $\{E[p^n]\}_n$ forms a $p$-divisible group)

(b) Galois group $G_{\mathbb{Q}}$ acts on the $\overline{Q}$-points preserving torsion

Thus you package this together and you get a continuous action of $G_{\mathbb{Q}}$ on the two-dimensional vector space $V_p(E) := \left(\lim_{\to n} E[p^n](\overline{Q}) \right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

On the other hand, $E[p^n](\overline{Q})$ is the covering group of the étale covering map $p^n : E \to E$. A simple argument using dual isogenies shows that $p^n$ is co-final among all covering maps $E' \to E$ of degree $p^n$. And thus $\text{Hom}(E[p^n](\overline{Q}), \mathbb{Z}/p^n\mathbb{Z}) = \text{Hom}(\pi_1^\text{ét}(E_{\overline{Q}}), \mathbb{Z}/p^n\mathbb{Z}) = H^1_\text{ét}(E_{\overline{Q}}, \mathbb{Z}/p^n\mathbb{Z})$.

Taking $\lim_{\to n}$ and tensoring with $\mathbb{Q}_p$ we see that $H^1_\text{ét}(E_{\overline{Q}}, \mathbb{Q}_p)$ and $V_p(A)$ are dual to each other.

2. Galois representations

The relationship between the étale cohomology and the other two are known, but there is a deeper idea of motives hiding in the background. One should be aware that each cohomology theory (including the ones for varying $p$) should be thought of as an avatar (Deligne uses the word “realization” in [2]) of a conjectural object $H^*(X)$ known as a motive$^3$. Moreover, part of the Tate conjecture is that the functor which sends a motive to its $p$-adic realization should be fully faithful (note: the same is not true when “motive” is replaced by “proper, smooth algebraic variety”). In this way one expects the étale cohomology groups to contain loads of information about the original variety $X$ (since, after all, $H^*(X)$ is meant to be its avatar). One simple example of this is the criterion of Néron, Ogg and Shafarevich.

2.1. The criterion of Neron-Ogg-Shafarevich. First, a word on Galois actions, and let’s stay in the realm of the group $G_{\mathbb{Q}}$. Suppose that $\ell$ is a prime. We can choose an extension $v_\ell$ of the $\ell$-adic valuation to $\overline{Q}$ and thus get decomposition groups $D_\ell := D(v_\ell)$ and inertia group $I_\ell := I(v_\ell)$ inside $G_{\mathbb{Q}}$. Removing the $v$ in $v_\ell$ is an abuse of notation since these are really only defined up to conjugacy in $G_{\mathbb{Q}}$. But do note that $D_\ell/I_\ell \cong \text{Gal}(\mathbb{F}_\ell/\mathbb{F}_\ell) \cong \hat{\mathbb{Z}}$ is pro-cyclic.

Definition. Suppose that $V$ is a vector space over a field $K$ equipped with a continuous $K$-linear action of $G_{\mathbb{Q}}$. We say that $V$ is unramified at $\ell$ if $V^{I_\ell} = V$, i.e. if $g \in I_\ell$ then $gv = v$ for all $v \in V$.

Note that although $I_\ell$ was only defined up to conjugacy, being unramified at $\ell$ is independent of this choice. As an example $\det(V_p(E)) = \chi_{\text{cycl}}$ is the $p$-adic cyclotomic character $G_{\mathbb{Q}} \to \text{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \cong \mathbb{Z}_p^\times$. Since $\mathbb{Q}(\mu_{p^\infty})$ is only ramified at $\ell = p$, we see

$$\det(V_p(E)) = \begin{cases} \text{unramified at } \ell & \text{if } \ell \neq p, \\ \text{ramified at } \ell & \text{if } \ell = p. \end{cases}$$

The following theorem tells us that $p$-adic cohomology can detect “good reduction” of abelian varieties.

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$^3$For more on this point of view, and especially Deligne’s paper, see the Fall 2014 STAGE seminar at MIT.
Theorem 1 (Néron-Ogg-Shafarevich). Let $\ell \neq p$ be two primes. If $E$ is an elliptic curve (or an abelian variety) over $\mathbb{Q}$ then the following are equivalent

(a) $E$ has good reduction at $\ell$,
(b) $H^1_{\text{ét}}(E, \mathbb{Q}_p)$ (or $V_p(E)$) is unramified at $\ell$

We’re not going to prove this theorem, but let us make some comments

(a) The meaning of good reduction is something we can understand for any variety $X$ over $\mathbb{Q}$. It means that there is a smooth proper $\mathbb{Z}_\ell$-scheme $X$ so that $X[1/\ell] \simeq X_{\mathbb{Q}_p}$. In the case where $X = E$ is an elliptic curve it means we can find a minimal Weierstrass equation over $\mathbb{Z}_\ell$ whose reduction modulo $\ell$ is non-singular.
(b) The condition (a) thus has no dependence on the prime $p$ and we deduce that “$V_p(E)$” is unramified at $\ell”$ is independent of $p \neq \ell$.
(c) Another corollary: good reduction is independent of isogeny class.
(d) This can be proven directly (with the Tate module, say) in [7] or the paper of Serre and Tate [6]. The implication (a) implies (b) is the shadow of a deep theorem: “smooth-proper base change” for étale cohomology (and thus is true for any variety with good reduction$^4$). The implication (b) implies (a) relies on the theory of Néron models, which are the “best” smooth models (which aren’t necessarily proper).
(e) Finally (this should’ve been earlier), the condition that $p \neq \ell$ is crucial as $V_p(E)$ is always ramified at $p$, regardless of good reduction or not (for example because $\det V_p(E)$ is the cyclotomic character).

Main question for this seminar:

Can you formulate a condition $\text{GR}(p)$ on $p$-adic Galois representations such that: if $E/\mathbb{Q}$ is an elliptic curve then $\text{GR}(p)$ holds for $V_p(E)$ if and only if $E$ has good reduction at $p$?

3. Plan for the rest of the seminar

There will be two halves to the semester.

3.1. $p$-divisible groups. The first step we will take is to study the general idea of $p$-divisible groups. This will take the next four weeks. There are three points to this.

- First, it is a classical paper and time meditating on it can only be a positive experience.
- Second, Grothendieck was able to (quickly after Tate’s paper I think) give a proxy to the question we asked in terms of $p$-divisible groups. That is, he formulated a condition $\text{GR}'(p)$ on $p$-divisible groups such that $\text{GR}'(p)$ holds for $\{E[p^n]\}_{n \geq 1}$ if and only if $E$ has good reduction. See [1, Theorem 7.1.13] and references contained there.
- Finally, an important precursor to answering the question is the study of the Hodge-Tate decomposition of $p$-adic étale cohomology, which is an analog of the Hodge decomposition we mentioned at the beginning, and even used to prove the Hodge decomposition.

The final point segways into the second half the seminar.

3.2. Fontaine’s theory (period rings). The overall goal of Fontaine’s theory is to extend scalars in cohomology to fruitfully study algebraic varieties. The Hodge-Tate decomposition mentioned above is the following theorem

Theorem 2 (Faltings, [4]). Suppose that $X/\mathbb{Q}_p$ is a proper smooth variety. Then

$$H^i_{\text{ét}}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \simeq \bigoplus_{0 \leq q \leq i} \mathbb{C}_p \cdot (\chi_{-q}^{\bullet})^{\otimes h_{i-q}}$$

$^4$Converses in general would be interesting. See the mathoverflow question 108953
where $h^{i-q,q} = \dim_{\mathbb{Q}_p} H^{i-q}(X, \Omega^q_{X/\mathbb{Q}_p})$ and the $\chi_p$ is the $p$-adic cyclotomic character again.

In particular, étale cohomology knows the Hodge numbers of the variety. We won’t study this theorem directly, but rather the second half of the semester will consist of

(a) Studying the (drastic) operation $V \mapsto V \otimes_{\mathbb{Q}_p} \mathbb{C}_p$ (Ax-Sen-Tate theory),

(b) Generalizing the period ring $\mathbb{C}_p$ to allow for more refined invariants of Galois representations. For example, the Hodge decomposition was really the associated graded of a filtration. Is there a filtration lurking behind the scenes of Faltings theorem?

(c) Answering the final question leads to the notion of “de Rham” representations (and crystalline and semistable). The semester will end with studying these classes of representations and hopefully explaining the answer to the question posed in the previous section.

References


