WEEK 6 NOTES

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Last week we covered the beginning of the theory of the Hodge-Tate decomposition for a \( p \)-divisible group. This week we will finish that result and clean up some other ideas from Tate’s paper.

0.1. The result. We let \( R \) be a complete discrete valuation ring with fraction field \( K \) and residue field \( k \). We let \( C = \overline{K} \) be the completion of an algebraic closure of \( K \), and \( O_C \) the ring of integers. We let \( \mathcal{G} = \text{Gal}(\overline{K}/K) \) act on \( C \) by continuity. Note that \( K \subset C \) is fixed by the action of \( \mathcal{G} \) and that if \( L \) is any \( K \subset L \subset \overline{K} \subset C \) then \( L^\mathcal{G} = K \) as well. Tate proved that there were no transcendental invariants.

We won’t discuss the proof of this today, and it is a more general theorem. In fact, let \( Z_p(1) = T(G_m(p)) \) be the Tate module of the \( p \)-divisible group \( G_m(p) \) and \( Z_p(-1) \) its dual \( Z_p[\mathcal{G}] \)-module. Write \( C_{(-1)} = C \otimes_{\mathbb{Z}_p} Z_p(\pm 1) \) with its natural \( C \)-semilinear action.

The parts of Tate’s theorem we need today are the following.

**Theorem** (Tate, [?], §3). 
\[(a) \ C^\mathcal{G} = K. \]
\[(b) \ C(\pm 1)^\mathcal{G} = (0) \text{ and } H^1(\mathcal{G}, C(\pm 1)) = (0) \]

Let’s now move onto our result. We will fix a \( p \)-divisible group \( G \) over \( R \) of height \( h \) and dimension \( n \). It’s dual group \( G^\vee \) has height \( h \) and dimension \( n \vee \). The relationship between them is \( n + n \vee = h \) [?, Proposition 3]. Let \( \mathcal{G} = \text{Gal}(\overline{K}/K) \).

Recall that we defined what the \( O_C \) points of \( G \) were by a funny formula

\[
G(O_C) = \lim_{\rightarrow i} \lim_{\rightarrow n} G_{p^n}(O_C/m^{n_i}_i O_C).
\]

The torsion subgroup of \( G(O_C)_{\text{tor}} \) is the “naïve” points \( \lim_{\rightarrow \overline{p}} G_{p^n}(O_C) \) and since \( C \) is algebraically closed (and thus \( G(O_C) \) is a \( p \)-divisible \( Z_p \)-module by [?], Corollary 2.4.2)) we have a short exact sequence

\[
0 \to G(O_C)_{\text{tor}} \to G(O_C) \xrightarrow{\log} t_G(C) \to 0.
\]

Note that the cokernel here is naturally an \( n \)-dimensional \( C \)-vector space. When \( G = G_m(p) \) this sequence is

\[
0 \to \mu_{p^n}(C) \to 1 + m_{O_C} \xrightarrow{\log} C \to 0.
\]

We saw in Ben’s talk how to construct the following diagram

\[
\begin{array}{ccccccccc}
0 & \to & G(O_C)_{\text{tor}} & \xrightarrow{\alpha} & G(O_C) & \xrightarrow{\alpha} & t_G(C) & \xrightarrow{d \alpha} & 0 \\
\downarrow{\alpha} & & \downarrow{\alpha} & & \downarrow{d \alpha} & & \downarrow{d \alpha} & & \downarrow{d \alpha} \\
0 & \to & \text{Hom}_{Z_p}(T(G^\vee), \mu_{p^n}(C)) & \to & \text{Hom}_{Z_p}(T(G^\vee), 1 + m_{O_C}) & \to & \text{Hom}_{Z_p}(T(G^\vee), C) & \to & 0.
\end{array}
\]

Everything in sight has actions of \( \mathcal{G} = \text{Gal}(\overline{K}/K) \), the actions on the bottom spaces are given by \((\sigma f)(x) = \sigma f(\sigma^{-1} x)\). The easiest examples are

**Example 1.** When \( G = G_m(p) \) then \( T(G^\vee) = Z_p \) with the trivial Galois action and the top and bottom rows are naturally isomorphism in each step.

When \( G = Q_p/Z_p \), then \( t_G(C) = (0) \), \( T(G^\vee) = Z_p(1) \) and the bottom row looks like, as Galois modules,

\[
0 \to \mu_{p^n}(-1) \to 1 + m_{O_C}(-1) \to C(-1) \to 0
\]

whereas the top row looks like

\[
0 \to Q_p/Z_p \xrightarrow{\cong} Q_p/Z_p \to 0.
\]

Thus \( \alpha \) and \( d \alpha \) are not isomorphisms in general.

In Ben’s talk we checked the following facts:
Remark. It’s good to keep the following thing in mind. The tangent space \( t_G(C) \) by definition has dimension \( n \), whereas the target of \( d\alpha \) has dimension \( h \) over \( C \). Thus, after knowing that \( d\alpha \) is injective, we see that \( \text{coker}(d\alpha) \) is a \( h - n = n^{\vee} \)-dimensional \( C \)-vector space. We’re almost ready to compute what it is, but a likely candidate is something doing with \( t_{G^{\vee}}(C) \).

It’s helpful to recall the key points from Ben’s talk. They were:

(a) Tate’s theorem above on the \( \mathcal{G} \)-invariants of \( C \) being \( K \). This implies that \( t_G(C)^{\mathcal{G}} = t_G(K) \).

(b) \( \mathcal{G} \) also implies \( G(R) = G(\mathcal{O}_C)^{\mathcal{G}} \). This point was glossed over in the previous talk but it is not quite formal. If \( G(\mathcal{O}_C) \) was the naïve points, it would be completely formal. Here is how you prove it for \( p \)-divisible groups:

- First handle the case where \( G \) is connected. In that case \( G(\mathcal{O}_C) = \mathcal{m}_C \) as a set, and also as a Galois module. Thus \( G(\mathcal{O}_C) = \mathcal{m}_C^{\mathcal{G}} = \mathcal{m}_R = G(R) \).
- Now consider étale \( G \). Then \( G(\mathcal{O}_C) = \varprojlim G_{p^n}(\mathcal{O}_C) \) is just the naïve points. The action commutes with the limits and thus \( G(\mathcal{O}_C)^{\mathcal{G}} = \varprojlim G_{p^n}(\mathcal{O}_C)^{\mathcal{G}} = \varprojlim G_{p^n}(R) = G(R) \).
- Now you take the \( \mathcal{O}_C \)-points of the connected étale sequence and after taking \( \mathcal{G} \)-invariants you get \( 0 \to G^{\alpha}(R) \to G(\mathcal{O}_C)^{\mathcal{G}} \to G^{\mathcal{G}}(R) \).

But \( G(R) \subset G(\mathcal{O}_C)^{\mathcal{G}} \) surjects onto \( G^{\mathcal{G}}(R) \) and thus so does \( G(\mathcal{O}_C)^{\mathcal{G}} \). Then it’s easy to see from the snake lemma that \( G(R) = G(\mathcal{O}_C)^{\mathcal{G}} \).

Having recalled this, the following theorem is at least well-posed.

**Theorem.** The map \( \alpha \) induces isomorphisms

\[
G(R) \xrightarrow{\cong} \Hom_{\mathcal{G}}(T(G^{\vee}), 1 + \mathcal{m}_C^{\mathcal{G}}) = \Hom_{\mathcal{G}}(T(G^{\vee}), 1 + \mathcal{m}_C)
\]

\[
t_G(K) \xrightarrow{\cong} \Hom_{\mathcal{G}}(T(G^{\vee}), C)^{\mathcal{G}} = \Hom_{\mathcal{G}}(T(G^{\vee}), C)
\]

There’s one corollary which is immediate from the theorem.

**Corollary 2.** \( \dim G \) is determined by \( T(G) \) as a \( \mathcal{G} \)-module.

**Proof of corollary.** \( T(G^{\vee})(-1) \) is the \( \mathcal{G} \)-dual of \( T(G) \) (see the proof of the theorem) and thus its enough to show \( T(G^{\vee}) \) determines \( \dim G \). But then the theorem says that \( \dim G \) is the dimension of the space \( (T(G^{\vee})^*)^{\mathcal{G}} \).

We note that Tate proved more than this in the end of his article. Indeed, he proved that \( G \mapsto T(G) \) was fully faithful \([?, \text{Theorem 4.2}]\). Let’s prove our theorem now.

**Proof of theorem.** We know that everything is injective, that was the result from last time. Let \( \alpha_R \) and \( d\alpha_R \) be the maps restricted to \( G(R) \), resp. \( t_G(K) \). We just explained why \( G(R) = G(\mathcal{O}_C)^{\mathcal{G}} \). Thus taking \( \mathcal{G} \)-fixed points of the original diagram we get

\[
0 \to G(R) \to \Hom_{\mathcal{G}}(T(G^{\vee}), 1 + \mathcal{m}_C) \to \text{coker}(\alpha)^{\mathcal{G}}.
\]

In particular, \( \text{coker}(\alpha_R) \subset \text{coker}(\alpha)^{\mathcal{G}} \). The same is true for \( \text{coker}(d\alpha_R) \). But \( \text{coker} \alpha = \text{coker} d\alpha \) so \( \text{coker}(\alpha_R) \subset \text{coker}(d\alpha_R) \). It suffices to show that \( \text{coker}(d\alpha_R) = \{0\} \).

This is useful: \( d\alpha_R \) is a \( K \)-linear map and we can try to count dimensions. Let’s clarify what we want to do. Let \( T(G^{\vee})^* = \Hom_{\mathcal{G}}(T(G^{\vee}), C) \) (and the same for \( G \)). This is a \( C \)-vector space whose dimension is \( h \) whereas \( t_G(K) \) has dimension \( n \). Let

\[
d^{\vee} = \dim_C(T(G^{\vee})^*)^{\mathcal{G}}
\]

\[
d = \dim_C(T(G)^*)^{\mathcal{G}}
\]

Since the image of \( d\alpha_R \) lands in the Galois equivariant \( \text{Hom} \), our goal is to show that \( n = d^{\vee} \). We know that \( n \leq d^{\vee} \) and \( n^{\vee} \leq d \) (for the same reason). We have the formula \( h = n + n^{\vee} \) and thus need to show that \( d^{\vee} + d \leq h \).
First let’s remember that we have a perfect pairing of Galois modules
\[ T(G) \times T(G^\vee) \to \mathbb{Z}_p(1). \]
This was explained last time. In particular, we have
\[ T(G)^* = \text{Hom}_{\mathbb{Z}_p}(T(G), C) \cong T(G^\vee) \otimes_{\mathbb{Z}_p} C(-1). \]
and ditto for \( G^\vee \). This gives us a pairing of Galois modules
\[ T(G)^* \times T(G^\vee)^* \to C(-1) \\
(x, y) \mapsto y(\iota(x)). \]
which is perfect.

Claim. \( (T(G)^*)^E \) and \( (T(G^\vee)^*)^E \) are orthogonal under this pairing.

Proof of claim. The image of elements paired like this lands in \( C(-1)^E = (0) \) (Tate’s theorem from the beginning). \( \square \)

With the claim in hand then we get that the pairing when restricted to \( (T(G)^*)^E \) factors
\[ (T(G)^*)^E \hookrightarrow \text{Hom}_C(T(G^\vee)^*, C(-1)) \]
\[ \text{Hom}_C(T(G^\vee)^*/(T(G^\vee)^*)^E, C(-1)) \]
The top left space has dimension \( d \), the bottom space has dimension \( h - d^\vee \). Thus \( d + d^\vee \leq h \), as we wanted to show. \( \square \)

Corollary 3. There is a naturally split decomposition of \( \mathcal{G} \)-modules
\[ T(G)^* = t_{G^\vee}(C) \oplus t_G(C)^*(-1). \]

Proof. We just showed that \( t_{G^\vee}(C) = (T(G)^*)^E \otimes_K C \hookrightarrow T(G)^* \). On the other hand, the pairing from the previous proof gives a surjective map
\[ T(G)^* = T(G^\vee) \otimes_{\mathbb{Z}_p} C(-1) \twoheadrightarrow t_G(C)^*(1). \]
Since \( t_G(C) = t_G(C)^E \otimes_K C \) also we have that the composition \( t_{G^\vee}(C) \to T(G)^* \to t_G(C)^*(1) \) is zero by the orthogonality claim in the previous proof. In particular the complex
\[ 0 \to t_{G^\vee}(C) \to T(G)^* \to t_G(C)^*(1) \to 0, \]
is exact: you just need to count dimensions. Finally, the sequence is split since \( H^1(\mathcal{G}, C(1)) = (0) \) and it is even naturally split since \( H^0(\mathcal{G}, C(1)) = (0) \). \( \square \)

1. Some waffle

I want to end by summarizing what has happened here, and in the previous lecture. Consider an elliptic curve \( E \) over \( R \) and its associated \( p \)-divisible group \( E[p^n] \).

Remember the terminology that
\[ E[p^n](\overline{k}) = \begin{cases} \mathbb{Z}/p^n \mathbb{Z} & \text{if } E \text{ is ordinary} \\ 0 & \text{if } E \text{ is supersingular}. \end{cases} \]
Since the connected-étale sequence commutes with base change \( \text{Spec } k \to \text{Spec } R \) (see the week one notes) we deduce that \( G^{\text{et}} \) has height zero or one, never two. This is all we need to know to understand the following table
The Tate module $T(E[p^\infty]) = T_p(E)$ is very familiar to us. For example we know that when $E$ is ordinary there is a one-dimensional quotient on which Galois acts trivially. The Hodge-Tate decomposition we just proved was that

$$T_p(E) \otimes \mathbb{Z}_p C = C \oplus C(1).$$

One might, rightfully, wonder if this is possibly happening before passing to $C$. Tate’s theorem(s) don’t seem to say anything about that, though what must be happening is that in the ordinary case, the inclusion $t_{E[p^\infty]}(K) \hookrightarrow (T(G)^* \otimes C)^\mathcal{G}$ is defined over $K$ actually, i.e. exists before passing to $C$.

You might also wonder about the Hodge-Tate decomposition versus the Hodge decomposition of the singular cohomology $H^1(E(C), C)$. Since the étale cohomology is dual to the Tate module it seem that the Hodge-Tate decomposition says

$$H^1_{\text{et}}(E(C), \mathbb{Q}_p) \otimes \mathbb{Q}_p C \simeq t_G(C) \oplus t_G(C)^*(-1)$$

as $C$-semilinear representations of $\mathcal{G}$. On the other hand, we have by Hodge theory

$$H^1(E(C), \mathbb{Q}_p) \otimes \mathbb{Q}_p C \simeq H^0(E(C, \Omega^1_{E/C}) \oplus H^1(E(C, \mathcal{O}_{E/C}).$$

And now you want to match these things up probably. It is easy to guess that $t_G(C)^*(-1) = H^0(E(C, \Omega^1_{E/C})$. The other identification is true as well but requires more ideas (and going back and really understanding the $p$-divisible groups associated to abelian schemes and the relation between Cartier duality and duality of abelian schemes).

### Table 1. Invariants associated to the elliptic curve

<table>
<thead>
<tr>
<th>$E$ is supersingular</th>
<th>$E$ is ordinary</th>
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</thead>
<tbody>
<tr>
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</tr>
<tr>
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<tr>
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