

We're going to talk about our new period ring  $B_{\text{dR}}$ . We will first talk about Witt vectors. We use the notation of Berger throughout (NOT the notation of Fontaine and Brinon-Conrad).

0.1. **Reminder on Witt vectors.**

**Definition.** A perfect ring of characteristic  $p$  is an  $\mathbf{F}_p$ -algebra such that  $x \mapsto x^p$  is bijective.

Standard examples are finite fields or algebraic extensions of finite fields.

**Definition.** A strict  $p$ -ring  $A$  is one that is complete and separated with respect to the  $p$ -adic topology and such that  $p$  is not a zero divisor and  $A/p$  is perfect. We call  $A/p$  the residue ring.

The completion of a function field in characteristic  $p$  is not a strict  $p$ -ring. Neither is the ring of integers in a ramified extension of  $\mathbf{Q}_p$ .

to each perfect ring  $R$  of characteristic  $p$  there exists a unique strict  $p$  ring  $W(R)$  such that  $W(R)/p = R$ .

What do we mean by unique? We mean unique up to unique isomorphism which induces the identity on the residue ring. So, if  $R \rightarrow S$  is a map of perfect rings of characteristic  $p$  we get a unique map  $W(R) \rightarrow W(S)$  making the diagram

$$\begin{array}{ccc} W(R) & \longrightarrow & W(S) \\ \downarrow & & \downarrow \\ R & \longrightarrow & S \end{array}$$

This explains why the ramified integer rings are ruled out.

**Example 1.** If  $R = \mathbf{F}_{p^n}$  then  $W(R)$  is the ring of integers in the unique degree  $n$  unramified extension of  $\mathbf{Q}_p$ . If  $R = \overline{\mathbf{F}}_p$ . Then  $W(R) = \mathcal{O}_{\widehat{\mathbf{Q}}_p^{\text{nr}}}$ .

There is a Teichmuller section: the residue map  $W(R) \rightarrow R$  has a multiplicative section  $[-] : R \rightarrow W(R)$ . In the case that  $R$  is a finite field, the section is the appropriate of roots of unity. Using the Teichmuller lift you can always write a general element as an infinite sum

$$\alpha = \sum_{i=0}^{\infty} [a_i] p^i, \quad a_i \in R.$$

0.2. **Moving on.**  $\mathcal{O}_C$  is not a strict  $p$ -ring because  $\mathcal{O}_C/p$  is not perfect. We're going to replace  $\mathcal{O}_C/p$  by a perfect ring and then apply a Witt vector construction. We formally make the  $p$ th power map an isomorphism

**Definition.**  $\tilde{\mathbf{E}}^+ = \varprojlim_{x \mapsto x^p} \mathcal{O}_C/p$ .

The  $p$ th power map of  $\mathcal{O}_C/p$  is a ring homomorphism and so  $\tilde{\mathbf{E}}^+$  is naturally a ring.

**Lemma 2.**  $\tilde{\mathbf{E}}^+$  is a perfect local domain of characteristic  $p$ .

*Proof.* It is characteristic  $p$ , ok. Let's show that its perfect. We have to show that  $x \mapsto x^p$  is surjective and injective.

surj: let  $(x_0, x_1, \dots) \in \tilde{\mathbf{E}}^+$ . Since  $x_i^p = x_{i-1}$  we get that  $(x_1, x_2, \dots)^p = (x_0, x_1, \dots)$ .

inj: The point is just the computation  $(p^{1/p}, p^{1/p^2}, \dots)^p = (0, p^{1/p}, p^{1/p^2}, \dots) \neq 0$ .

Let's check that it's a domain. Let  $(x_i), (y_i) \in \tilde{\mathbf{E}}^+$  such that  $(x_i y_i) = 0$  in  $\tilde{\mathbf{E}}^+$ . Thus  $x_i y_i \equiv 0 \pmod{p\mathcal{O}_C}$ . Choose lifts  $x'_i$  and  $y'_i$  in  $\mathcal{O}_C$ . For  $i$  sufficiently large,  $v_p(x'_i) < 1$  and  $v_p(y'_i) < 1$ . Extracting  $p$ th roots we can then assume that  $v_p(x'_i y'_i) < 1$  as well. Thus  $x_i y_i \neq 0$  in  $\mathcal{O}_C/p\mathcal{O}_C$ .

Why is it local? There is a map  $\tilde{\mathbf{E}}^+ = \mathcal{O}_C/p \rightarrow \overline{\mathbf{F}}_p$ . This gives you a maximal ideal  $\tilde{\mathbf{E}}^+$ . In order to show it's local you have to show the complement of the kernel is entirely units. So if  $x \notin \ker$  then  $x = (x_0, x_1, \dots)$  such that  $x_0 \in (\mathcal{O}_C/p)^\times$ . Extracting  $p$ th roots, all of  $x_i \in (\mathcal{O}_C/p)^\times$ .  $\square$

It turns out that we can get a (multiplicative) map  $\theta_0 : \tilde{\mathbf{E}}^+ \rightarrow \mathcal{O}_C$  such that

$$\begin{array}{ccc} \tilde{\mathbf{E}}^+ & \xrightarrow{\theta_0} & \mathcal{O}_C \\ & \searrow & \swarrow \\ & \mathcal{O}_C/p & \end{array}$$

This is defined as follows. If  $x = (x_i) \in \tilde{\mathbf{E}}^+$  then choose lifts  $x'_i \in \mathcal{O}_C$ . We then define

$$\theta(x) = \lim_{j \rightarrow \infty} (x'_j)^{p^j}.$$

It turns out that this is independent of the choice of lifts  $\{x'_j\}$ .

We let  $\tilde{\mathbf{A}}^+ = W(\tilde{\mathbf{E}}^+)$ . This falls under our construction of Witt vectors before.

But the funny thing is that we have a diagram

$$\begin{array}{ccc} \tilde{\mathbf{A}}^+ & \xrightarrow{\theta} & \mathcal{O}_C \\ \downarrow & \nearrow \theta_0 & \downarrow \\ \tilde{\mathbf{E}}^+ & \longrightarrow & \mathcal{O}_C/p \end{array}$$

Note that although  $\theta_0$  on  $\tilde{\mathbf{E}}^+$  is *not a ring homomorphism* the map  $\theta$  on  $\tilde{\mathbf{A}}^+$  is *a ring homomorphism*. The diagram *does not commute* since  $\theta(p) = p$ . The map  $\theta$  is given on Teichmüller representatives as  $\theta([x]) = \theta_0(x)$ .

Now let  $\tilde{\mathbf{B}}^+ = \tilde{\mathbf{A}}^+[1/p]$ . Then  $\theta : \tilde{\mathbf{B}}^+ \rightarrow C$  because  $p$  is invertible in  $C$ . Moreover,  $\tilde{\mathbf{B}}^+$  is a discrete valuation ring whose residue field is  $C$ .

**Example 3.** There are two important elements of  $\tilde{\mathbf{E}}^+$ . We let

$$\varpi = (0, p^{1/p}, p^{1/p^2}, \dots)$$

(this depends on a choice). We can take a Teichmüller representative  $[\varpi] \in \tilde{\mathbf{A}}^+$ . Moreover,  $\theta([\varpi]) = p$  and  $v_{\tilde{\mathbf{E}}^+}(\varpi) = 1$ .

The other interesting element is the following  $(1 = \varepsilon^{(0)}, \varepsilon^{(1)}, \dots)$  compatible system of  $p^n$ th roots of unity in  $\mathcal{O}_C$ . This is the  $p$ -adic analog of an orientation in complex analysis. Having chosen this we get  $\varepsilon \in \tilde{\mathbf{E}}^+$ . Note that  $\varepsilon - 1$  is in the maximal ideal and  $v(\varepsilon - 1) = \frac{p}{p-1}$ .

**0.3. Another description of  $\tilde{\mathbf{E}}^+$ .** Recall that  $\tilde{\mathbf{E}}^+ = \varprojlim \mathcal{O}_C/p$ . We write elements as  $(x_0, x_1, \dots)$

**Proposition 4.** *There exists a bijective map  $\varprojlim_{x \mapsto x^p} \mathcal{O}_C \rightarrow \tilde{\mathbf{E}}^+$ .*

The fact that this map is bijective underlies many of the claims already discussed.

*Proof.* Write  $(x^{(0)}, x^{(1)}, x^{(2)}, \dots)$  for elements of the left hand side. Note that the map

$$\varprojlim \mathcal{O}_C \rightarrow \mathcal{O}_C$$

by  $(x^{(0)}, x^{(1)}, x^{(2)}, \dots) \mapsto x^{(0)}$  is surjective. We now define

$$\begin{aligned} \varprojlim \mathcal{O}_C &\rightarrow \varprojlim \mathcal{O}_C/p \\ (x^{(0)}, x^{(1)}, \dots) &\mapsto (\bar{x}^{(0)}, \bar{x}^{(1)}, \dots) \end{aligned}$$

This is the claimed map. Let's check it is bijective.

inj: Suppose that  $x^{(i)} \equiv y^{(i)} \pmod p$  for all  $i$ . Then  $x^{(i+j)} \equiv y^{(i+j)} \pmod p$  and thus  $x^{(i)} \equiv y^{(i)} \pmod{p^j}$  for all  $j$ . Thus  $x^{(i)} = y^{(i)}$ .

surj: Suppose that  $(x_0, x_1, \dots) \in \widetilde{\mathbf{E}}^+$ . Choose lifts  $x'_0, x'_1, \dots$  of  $x_0, x_1, \dots$ . We need to make these strictly compatible under the  $p$ th power map. To do this, we let

$$x^{(0)} = \lim_{j \rightarrow \infty} (x'_j)^{p^j}$$

This limit converges and is independent of the lifts  $x'_j$ . If we want to define  $x^{(i)}$  then we let

$$x^{(i)} = \varprojlim_{j \rightarrow \infty} (x'_j)^{p^{j-i}}$$

□

Now, using the ring structure on  $\widetilde{\mathbf{E}}^+$  we can put a ring structure on  $\varprojlim \mathcal{O}_C$ . Let's compute out what addition is. Suppose that  $(x^{(0)}, x^{(1)}, \dots) \in \widetilde{\mathbf{E}}^+$  and  $(y^{(0)}, y^{(1)}, \dots)$  the same. Let  $(x_0, x_1, \dots)$  and  $(y_0, y_1, \dots)$  be their reduction mod  $p$ . We then look at the addition

$$(x_0 + y_0, x_1 + y_1, x_2 + y_2, \dots) \in \widetilde{\mathbf{E}}^+.$$

We then get

$$(x + y)^{(i)} = \lim_{j \rightarrow \infty} \underbrace{((x_{i+j} + y_{i+j})')^{p^j}}_{\text{lift}}.$$

We don't actually have to choose lifts though because we're given lifts already. Thus we get

$$(x + y)^{(i)} = \lim_{j \rightarrow \infty} (x^{(i+j)} + y^{(i+j)})^{p^j}.$$

We use this to define the valuation on  $\widetilde{\mathbf{E}}^+$  given by  $v(x) = v_p(x^{(0)})$ . Let  $\varepsilon$  be the  $p$ th power sequence of primitive  $p$ th roots of unity. What is  $\varepsilon - 1$ ? We get

$$(\varepsilon - 1)^{(i)} = \lim_{j \rightarrow \infty} (\varepsilon^{(i+j)} - 1)^{p^j}.$$